5.2 Exploring Translations of Quadratic Relations

GOAL

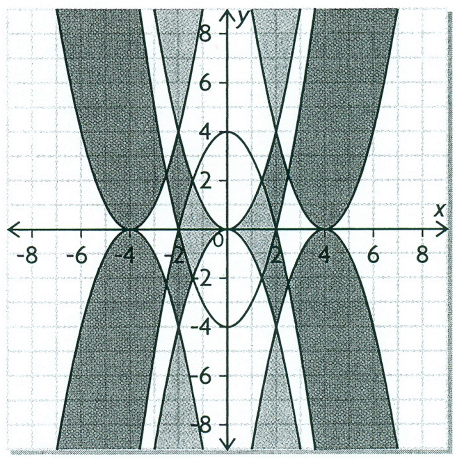
Investigate the roles of h and k in the graphs of $y = x^2 + k$, $y = (x - h)^2$, and $y = (x - h)^2 + k$.

YOU WILL NEED

- grid paper
- ruler
- graphing calculator

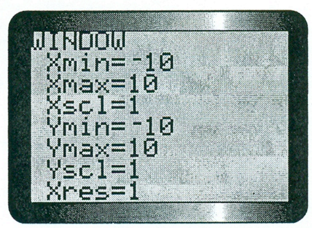
EXPLORE the Math

Hammad has been asked to paint a mural of overlapping parabolas on a wall in his school. A sketch of his final design is shown at the right. He is using his graphing calculator to try to duplicate his design. His design uses parabolas that have the same shape as $y = x^2$, but he doesn't know what equations he should enter into his graphing calculator to place the parabolas in different locations on the screen.



? What is the connection between the location of the vertex of a parabola and the equation of its quadratic relation?

A. Enter the equation $y = x^2$ as Y1 in the equation editor of a graphing calculator. Graph the equation using the window settings shown.



B. Enter an equation of the form $y = x^2 + k$ in Y2 by adding or subtracting a number after the x^2 term. For example, $y = x^2 + 1$ or $y = x^2 - 3$. Graph your equation, and compare the graph with the graph of $y = x^2$. Try several other equations, replacing the one you have in Y2 each time. Be sure to change the number you add or subtract after the x^2 term.

C. Copy this table. Use the table to record your findings for part B.

Value of k	Equation	Distance and Direction from $y = x^2$	Vertex
0	$y = x^2$	not applicable	(0, 0)

Tech *Support*

For help graphing relations, changing window settings, and tracing along a graph using a TI-83/84 graphing calculator, see Appendix B-2 and B-4. If you are using a TI-*n*spire, see Appendix B-38 and B-40.

Tech *Support*

Use the TRACE key and the up arrow ▲ to help you distinguish one graph from another.

D. Investigate what happens to the graph of $y = x^2$ when a number is added to or subtracted from the value of x before it is squared, creating an equation of the form $y = (x - h)^2$. For example, $y = (x + 1)^2$ or $y = (x - 2)^2$. Graph your new equations in Y2 each time using a graphing calculator. Then copy this table and record your findings.

Value of h	Equation	Distance and Direction from $y = x^2$	Vertex
0	$y = x^2$	not applicable	(0, 0)

E. Identify the type of transformations that have been applied to the graph of $y = x^2$ to obtain the graphs in your table for part C and your table for part D.

F. Make a conjecture about how you could predict the equation of a parabola if you knew the translations that were applied to the graph of $y = x^2$.

G. Copy and complete this table to investigate and test your conjecture for part F.

Value of h	Value of k	Equation	Relationship to $y = x^2$		Vertex
			Left/Right	Up/Down	
0	0	$y = x^2$	not applicable	not applicable	(0, 0)
			left 3	down 5	
4	1	$y = (x - 4)^2 + 1$			
					(−2, 6)
		$y = (x + 5)^2 - 3$			

H. Use what you have discovered to identify the equations that Hammad should type into his calculator to graph the parabolas in the mural design.

I. If the equation of a quadratic relation is given in the form $y = (x - h)^2 + k$, what can you conclude about its vertex?

Reflecting

J. Describe how changing the value of k in $y = x^2 + k$ affects
i) the graph of $y = x^2$
ii) the coordinates of each point on the parabola $y = x^2$
iii) the parabola's vertex and axis of symmetry

K. Describe how changing the value of h in $y = (x - h)^2$ affects
i) the graph of $y = x^2$
ii) the coordinates of each point on the parabola $y = x^2$
iii) the parabola's vertex and axis of symmetry

L. For parabolas defined by $y = (x - h)^2 + k$,

i) how do their shapes compare to the parabola defined by $y = x^2$?

ii) what is the equation of the axis of symmetry?

iii) what are the coordinates of the vertex?

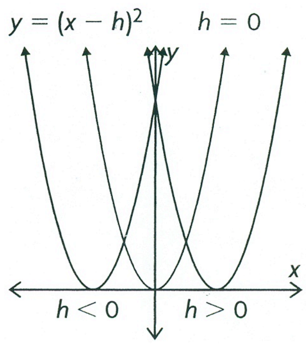
In Summary

Key Ideas

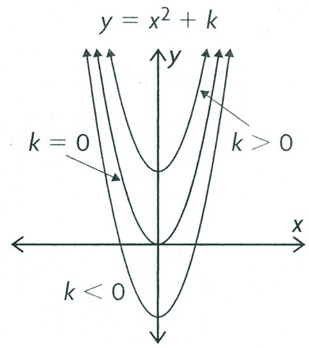
- The graph of $y = (x - h)^2 + k$ is congruent to the graph of $y = x^2$, but translated horizontally and vertically.
- Translations can also be described as shifts. Vertical shifts are up or down, and horizontal shifts are left or right.

Need to Know

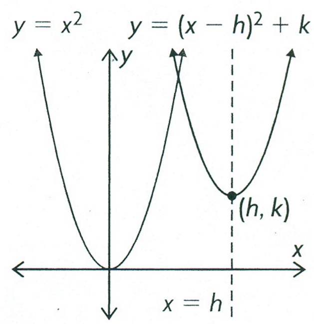
- The value of h tells how far and in what direction the parabola is translated horizontally. If $h < 0$, the parabola is translated h units left. If $h > 0$, the parabola is translated h units right.
- The vertex of $y = (x - h)^2$ is the point $(h, 0)$.
- The equation of the axis of symmetry of $y = (x - h)^2$ is $x = h$.



- The value of k tells how far and in what direction the parabola is translated vertically. If $k < 0$, the parabola is translated k units down. If $k > 0$, the parabola is translated k units up.
- The vertex of $y = x^2 + k$ is the point $(0, k)$.
- The equation of the axis of symmetry of $y = x^2 + k$ is $x = 0$.



- The vertex of $y = (x - h)^2 + k$ is the point (h, k).
- The equation of the axis of symmetry of $y = (x - h)^2 + k$ is $x = h$.



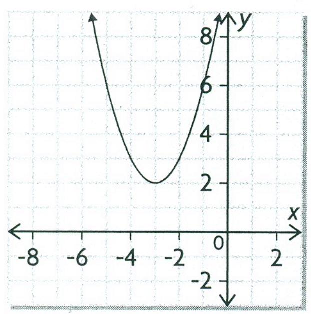
FURTHER Your Understanding

1. The following transformations are applied to a parabola with the equation $y = x^2$. Determine the values of h and k, and write the equation in the form $y = (x - h)^2 + k$.

a) The parabola moves 3 units right.
b) The parabola moves 4 units down.
c) The parabola moves 2 units left.
d) The parabola moves 5 units up.
e) The parabola moves 7 units down and 6 units left.
f) The parabola moves 2 units right and 5 units up.

2. Match each equation with the correct graph.

a) $y = (x - 2)^2 + 3$ **c)** $y = (x + 3)^2 - 2$
b) $y = (x + 2)^2 - 3$ **d)** $y = (x - 3)^2 + 2$

i)


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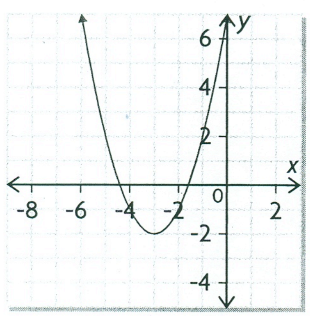

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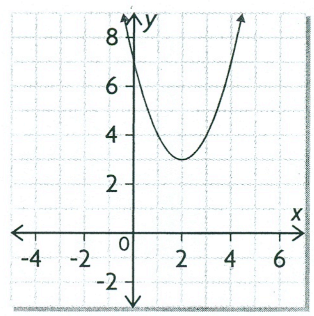

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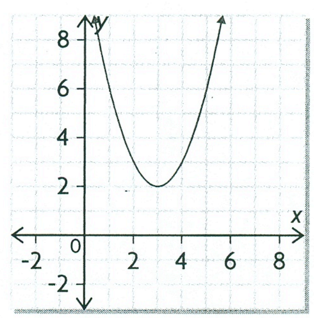

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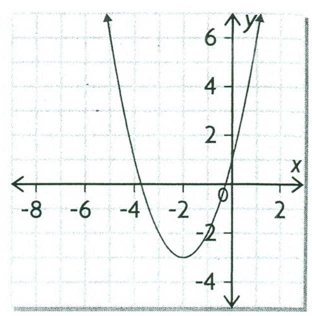

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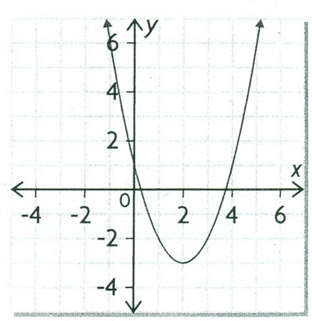


3. Sketch the graph of each relation by hand. Start with the graph of $y = x^2$, and apply the appropriate transformations.

a) $y = x^2 - 4$ **c)** $y = x^2 + 2$ **e)** $y = (x + 1)^2 - 2$
b) $y = (x - 3)^2$ **d)** $y = (x + 5)^2$ **f)** $y = (x - 5)^2 + 3$

4. Describe the transformations that are applied to the graph of $y = x^2$ to obtain the graph of each quadratic relation.

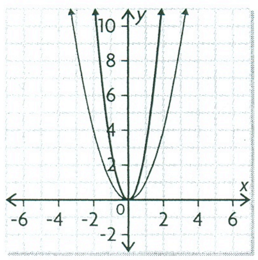
a) $y = x^2 + 5$ **c)** $y = -3x^2$ **e)** $y = \frac{1}{2}x^2$
b) $y = (x - 3)^2$ **d)** $y = (x + 7)^2$ **f)** $y = (x + 6)^2 + 12$

5. State the vertex and the axis of symmetry of each parabola in question 4.

11. Answers may vary, e.g.,

Definition: A relation that can be described by an equation with a polynomial whose highest degree is 2	Special Properties: The graph has a vertical line of symmetry. The graph also has a single minimum or maximum value.
Quadratic Relation	
Examples: $y = x^2 + 9x + 2$; $y = 2(x + 4)(x - 6)$; $y = 4(x + 2)^2 - 3$	Non-examples: $y = x + 9$; $y = x^3 + 9x + 3$; $y = \sqrt{x}$

Lesson 5.1, page 256

1. a) iv b) iii c) i d) ii
2. a) (1, 5) b) (−2, −12) c) (5, −15) d) (−4, 8)
3. a) Answers may vary, e.g., $y = 4x^2$; $y = 1.01x^2$
   b) Answers may vary, e.g., $y = 0.5x^2$; $y = -0.1x^2$
   c) Answers may vary, e.g., $y = -3.1x^2$; $y = -6x^2$
4. a)


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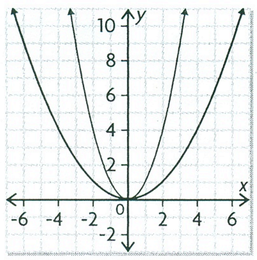

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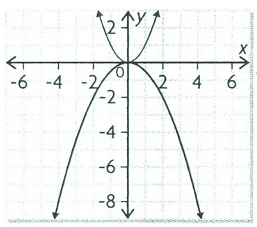

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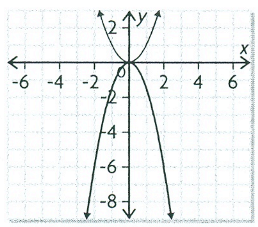

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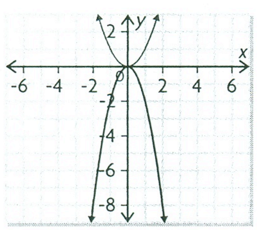

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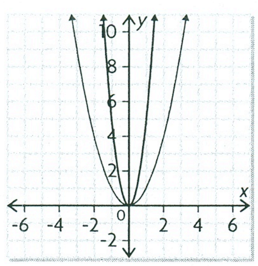


5. a) vertical stretch by a factor of 4; $y = 4x^2$
   b) vertical compression by a factor of $\frac{1}{2}$, reflected in the x-axis; $y = -\frac{1}{2}x^2$
   c) vertical stretch by a factor of 2.5, reflected in the x-axis; $y = -2.5x^2$
   d) vertical compression by a factor of $\frac{1}{4}$; $y = \frac{1}{4}x^2$
6. Choose the point (2, −0.5), and substitute this point into $y = ax^2$; solve for a; Answers may vary, e.g., $y = -0.125x^2$.
7. a) Answers may vary, e.g., $y = -\frac{5}{9}x^2$
   b) Answers may vary, e.g., $y = -\frac{3}{16}x^2$
8. a) vertical stretch by a factor of 4; (2, 16)
   b) reflection in the x-axis, vertical compression by a factor of $\frac{2}{3}$; $\left(2, -\frac{8}{3}\right)$
   c) vertical compression by a factor of 0.25; (2, 1)
   d) reflection in the x-axis, vertical stretch by a factor of 5; (2, −20)
   e) reflection in the x-axis; (2, −4)
   f) vertical compression by a factor of $\frac{1}{5}$; $\left(2, \frac{4}{5}\right)$
9. Answers may vary, e.g., $y = -\frac{1}{9}x^2$
10. Disagree. Changing the value of a to 1 or −1 will make $y = ax^2$ congruent to $y = x^2$.
11.

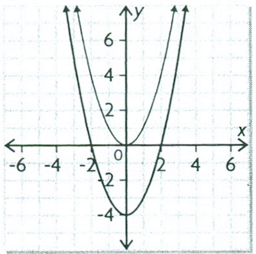
Equation	Direction of Opening (upward/ downward)	Description of Transformation (stretch/compress)	Shape of Graph Compared with Graph of $y = x^2$ (wider/narrower)
$y = 5x^2$	upward	stretch	narrower
$y = 0.25x^2$	upward	compress	wider
$y = -\frac{1}{3}x^2$	downward	compress	wider
$y = -8x^2$	downward	stretch	narrower

12. a) All the y-coordinates are multiplied by a negative number. This means that all the points on the graph $y = ax^2$ are reflected in the x-axis, causing the parabola to open downward.
    b) The y-coordinates of the points on the graph are multiplied by a fraction whose magnitude is less than 1, so the points are moved toward the x-axis, making the parabola wider.
    c) Since the y-coordinate of the vertex is 0, and multiplying 0 by any number results in a value of 0, the vertex is not affected.
13. It has the same effect on all graphs.
14. a) As the value of a increases, the radius of the circle decreases. As the value of a decreases, the radius of the circle increases.
    b) The graph of $ax^2 + by^2 = r^2$ is a circle that has been stretched or compressed both horizontally and vertically for all values of a and b, where $a \neq 1$ and $b \neq 1$. The resulting oval shape is called an ellipse. As the value of a increases, the width of the oval shape along the x-axis decreases. As the value of a decreases, the width of the oval shape along the x-axis increases. As the value of b increases, the width of the oval shape along the y-axis decreases. As the value of b decreases, the width of the oval shape along the y-axis increases.

Lesson 5.2, page 262

1. a) $h = 3$, $k = 0$; $y = (x - 3)^2$
   b) $h = 0$, $k = -4$; $y = x^2 - 4$
   c) $h = -2$, $k = 0$; $y = (x + 2)^2$
   d) $h = 0$, $k = 5$; $y = x^2 + 5$
   e) $h = -6$, $k = -7$; $y = (x + 6)^2 - 7$
   f) $h = 2$, $k = 5$; $y = (x - 2)^2 + 5$
2. a) iii b) v c) ii d) iv

Answers

3. a)


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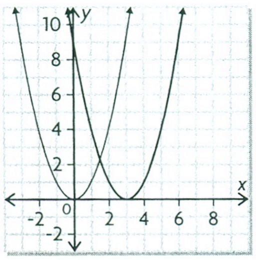

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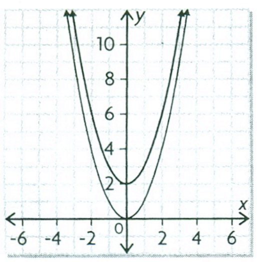

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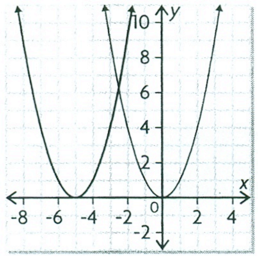

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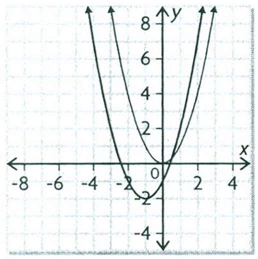

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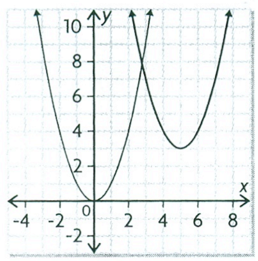
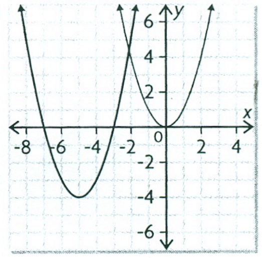


4. a) The parabola moves 5 units up.
   b) The parabola moves 3 units right.
   c) The parabola is reflected in the x-axis and vertically stretched by a factor of 3.
   d) The parabola moves 7 units left.
   e) The parabola is vertically compressed by a factor of $\frac{1}{2}$.
   f) The parabola moves 6 units left and 12 units up.
5. a) vertex: (0, 5); equation of the axis of symmetry: $x = 0$
   b) vertex: (3, 0); equation of the axis of symmetry: $x = 3$
   c) vertex: (0, 0); equation of the axis of symmetry: $x = 0$
   d) vertex: $(-7, 0)$; equation of the axis of symmetry: $x = -7$
   e) vertex: (0, 0); equation of the axis of symmetry: $x = 0$
   f) vertex: $(-6, 12)$; equation of the axis of symmetry: $x = -6$

Lesson 5.3, page 269

1. a) translation 3 units down
   b) translation 5 units left
   c) vertical compression by a factor of $\frac{1}{2}$, reflection in the x-axis
   d) vertical stretch by a factor of 4, translation 2 units left and 16 units down
2. a) i) upward ii) $(0, -3)$ iii) $x = 0$
   b) i) upward ii) $(-5, 0)$ iii) $x = -5$
   c) i) downward ii) $(0, 0)$ iii) $x = 0$
   d) i) upward ii) $(-2, -16)$ iii) $x = -2$

3. a)


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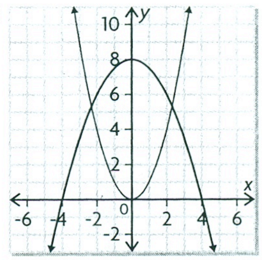

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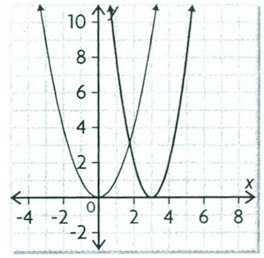

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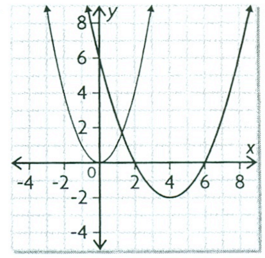
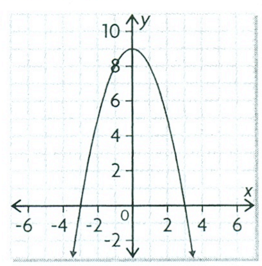


4. a) reflection in the x-axis, translation 9 units up
   b) translation 3 units right
   c) translation 2 units left and 1 unit down
   d) reflection in the x-axis, translation 6 units down
   e) vertical stretch by a factor of 2, reflection in the x-axis, translation 4 units right and 16 units up
   f) vertical compression by a factor of $\frac{1}{2}$, translation 6 units left and 12 units up
   g) vertical compression by a factor of $\frac{1}{2}$, reflection in the x-axis, translation 4 units left and 7 units down
   h) vertical stretch by a factor of 5, translation 4 units right and 12 units down

5. a)


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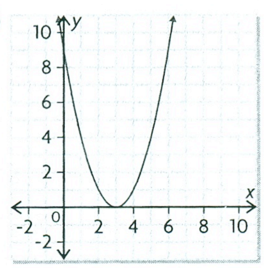

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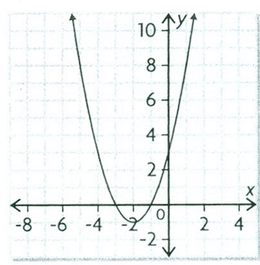

d)