5.1 Stretching/Reflecting Quadratic Relations

YOU WILL NEED

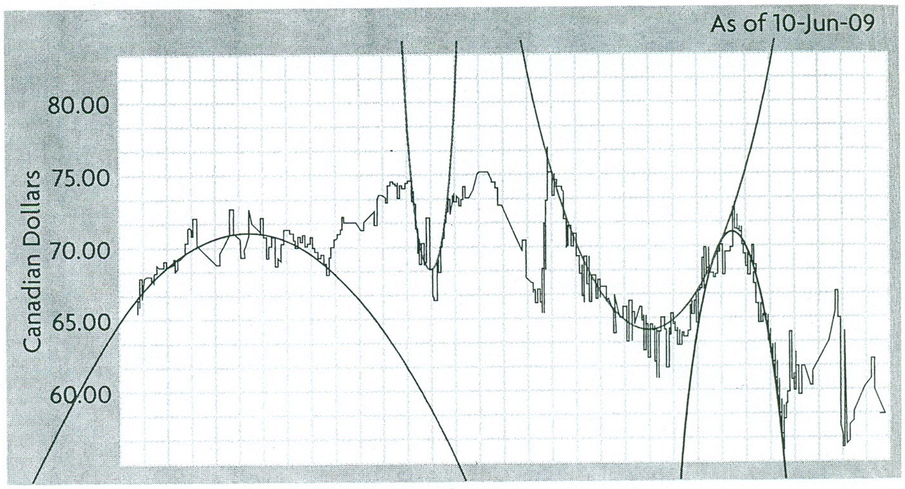
- graphing calculator
- dynamic geometry software, or grid paper and ruler

GOAL

Examine the effect of the parameter a in the equation $y = ax^2$ on the graph of the equation.

INVESTIGATE the Math

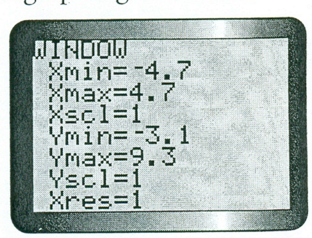
Suzanne's mother checks the family's investments regularly. When Suzanne saw the stock chart that her mother was checking, she noticed trends in sections of the graph. These trends looked like the shapes of the parabolas she had been studying. Each "parabola" was a different shape.



? What is the relationship between the value of a in the equation $y = ax^2$ and the shape of the graph of the relation?

A. Enter $y = x^2$ as Y1 in the equation editor of a graphing calculator.

B. The window settings shown are "friendly" because they allow you to trace using intervals of 0.1. Graph the parabola using these settings.



C. Enter $y = 2x^2$ in Y2 and $y = 5x^2$ in Y3, and graph these quadratic relations. What appears to be happening to the shape of the graph as the value of a increases?

Tech | *Support*

For help graphing relations and adjusting the window settings using a TI-83/84 graphing calculator, see Appendix B-2 and B-4. If you are using a TI-*n*spire, see Appendix B-38 and B-40.

D. Where would you expect the graph of $y = 3x^2$ to appear, relative to the other three graphs? Check by entering $y = 3x^2$ into Y4 and graph with a thick line. Was your conjecture correct?

E. Where would you expect the graphs of $y = \frac{1}{2}x^2$ and $y = \frac{1}{4}x^2$ to appear, relative to the graph of $y = x^2$? Clear the equations from Y2, Y3, and Y4. Enter $y = \frac{1}{2}x^2$ into Y2 and $y = \frac{1}{4}x^2$ into Y3, and graph these quadratic relations. Describe the effect of the **parameter** a on the parabola when $0 < a < 1$.

F. Where you would expect the graph of $y = \frac{3}{4}x^2$ to appear, relative to the other three graphs? Check by entering $y = \frac{3}{4}x^2$ into Y4 and graph with a thick line.

G. Clear the equations from Y2, Y3, and Y4. Enter $y = -4x^2$ into Y2 and $y = -\frac{1}{4}x^2$ into Y3, and graph these quadratic relations. Describe the effect of a on the parabola when $a < 0$.

H. Ask a classmate to give you an equation in the form $y = ax^2$, where $a < 0$. Describe to your classmate what its graph would look like relative to the other three graphs. Verify your description by graphing the equation in Y4.

I. How does changing the value of a in the equation $y = ax^2$ affect the shape of the graph?

Reflecting

J. Which parabola in the stock chart has the greatest value of a? Which has the least value of a? Which parabolas have negative values of a? Explain how you know.

K. What happens to the x-coordinates of all the points on the graph of $y = x^2$ when the parameter a is changed in $y = ax^2$? What happens to the y-coordinates? What happens to the shape of the parabola near its vertex?

L. State the ranges of values of a that will cause the graph of $y = x^2$ to be
i) **vertically stretched**
ii) **vertically compressed**
iii) reflected across the x-axis

Tech *Support*

Move the cursor to the left of Y4. Press ENTER to change the line style to make the line thick.

parameter

a coefficient that can be changed in a relation; for example, a, b, and c are parameters in $y = ax^2 + bx + c$

vertical stretch

a transformation that increases all the y-coordinates of a relation by the same factor

vertical compression

a transformation that decreases all the y-coordinates of a relation by the same factor

APPLY the Math

EXAMPLE 1 Selecting a transformation strategy to graph a parabola

a) Sketch the graph of the equation $y = 3x^2$ by transforming the graph of $y = x^2$.

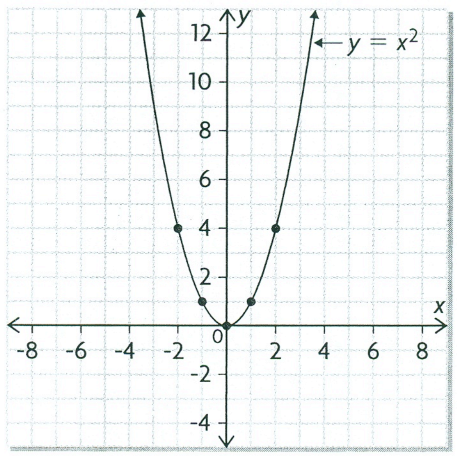
b) Describe how the graphs of $y = 3x^2$ and $y = -3x^2$ are related.

Zack's Solution

a)

x	−2	−1	0	1	2
y	4	1	0	1	4

I created a table of values to determine five points on the graph of $y = x^2$.



I plotted the points on a grid and joined them with a smooth curve.

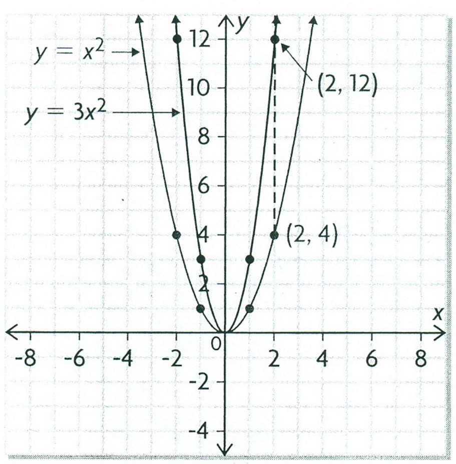
I can use these five points any time I want to sketch the graph of $y = x^2$ because they include the vertex and two points on each side of the parabola.

I decided to call this my five-point sketch.

x	−2	−1	0	1	2
y	12	3	0	3	12

To transform my graph into a graph of $y = \mathbf{3}x^2$, I multiplied the y-coordinates of each point on $y = x^2$ by **3**. For example,

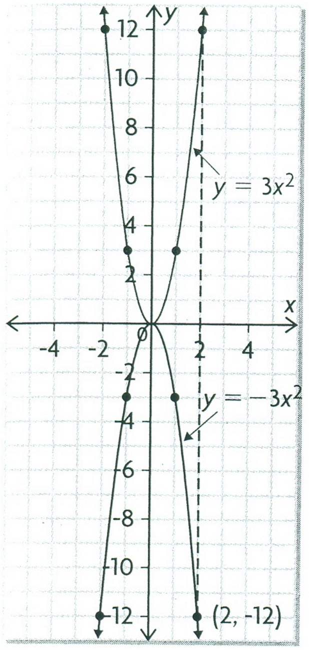
(2, 4) → 4 × **3** → (2, **12**)



I plotted and joined my new points to get the graph of $y = 3x^2$. $a = 3$ represents a vertical stretch by a factor of 3. This means that the y-coordinates of the points on the graph of $y = 3x^2$ will become greater faster, so the parabola will be narrower near its vertex compared to the graph of $y = x^2$.

b)

x	−2	−1	0	1	2
y	−12	−3	0	−3	−12



The graph of $y = -3x^2$ is the reflection of the graph of $y = 3x^2$ in the x-axis.

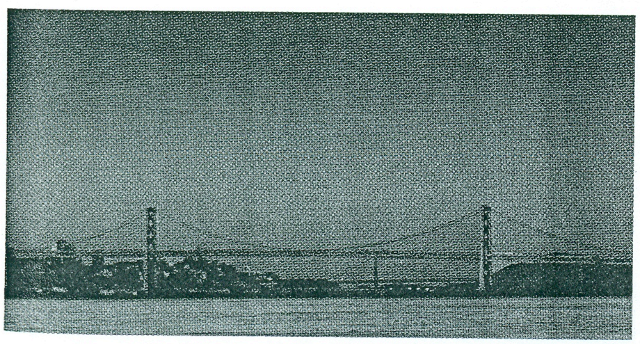
To get the graph of $y = -3x^2$, I multiplied the y-coordinates of all the points on the graph of $y = 3x^2$ by $\mathbf{-1}$. For example,

(2,12) → $12 \times (\mathbf{-1})$ → (2,$\mathbf{-12}$)

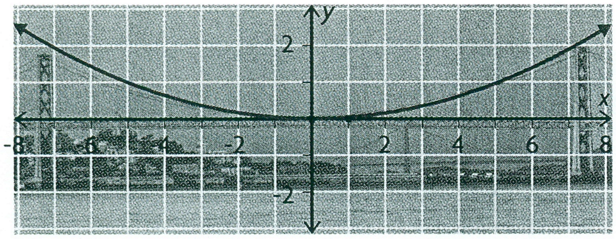
$a = -3$ represents a vertical stretch by a factor of 3 and a reflection in the x-axis. This means that all the points on the graph of $y = 3x^2$ are reflected in the x-axis.

EXAMPLE 2 Connecting the value of a to a graph

Determine an equation of a quadratic relation that models the arch of San Francisco's Bay Bridge in the photograph below.



Mary's Solution: Representing the picture on a hand-drawn grid



I used a photocopy of the photograph. I laid a transparent grid with axes on top of the photocopy.

I placed the origin at the vertex of the arch. I did this since all parabolas defined by $y = ax^2$ have their vertex at (0, 0).

I located a point on the graph and estimated the coordinates of the point to be (5, 1).

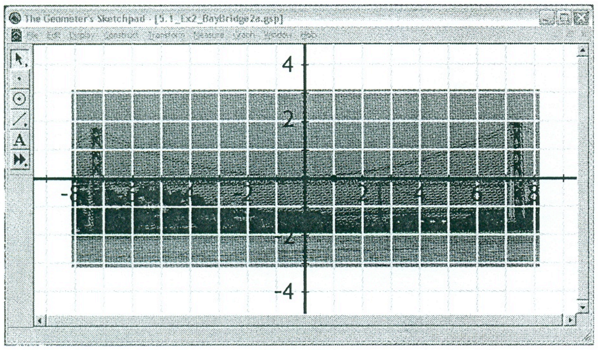
$$y = ax^2$$
$$1 = a(5)^2$$
$$1 = 25a$$
$$\frac{1}{25} = \frac{25a}{25}$$
$$\frac{1}{25} = a$$

The equation of the graph is in the form $y = ax^2$. To determine the value of a, I had to determine the coordinates of a point on the parabola. I chose the point (5, 1). I substituted $x = 5$ and $y = 1$ into the equation and solved for a.

An equation that models the arch of the bridge is $y = \frac{1}{25}x^2$.

The graph that models the arch is a vertical compression of the graph of $y = x^2$ by a factor of $\frac{1}{25}$.

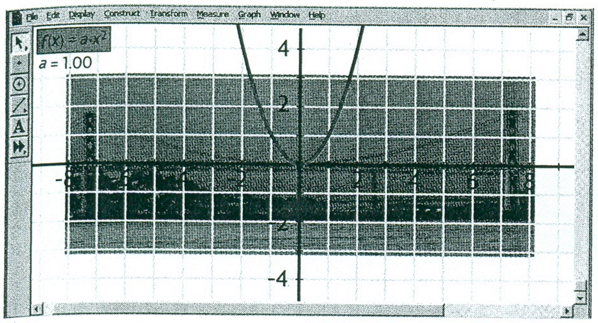
Sandeep's Solution: Selecting dynamic geometry software



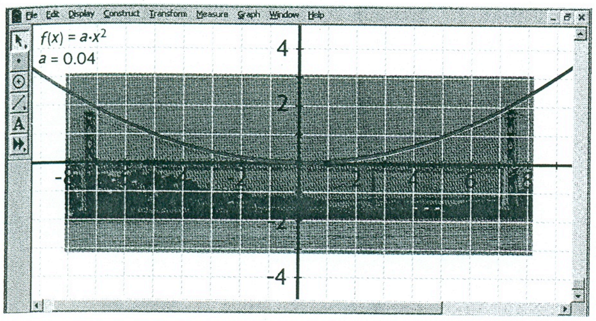
I imported the photograph into dynamic geometry software. I superimposed a grid over the photograph. Then I adjusted the grid so that the origin was at the vertex of the bridge's parabolic arch. I need to create a graph using the relation $y = ax^2$ by choosing a value for the parameter a.

Tech *Support*

For help creating and graphing relations using parameters in dynamic geometry software, as well as animating the parameter, see Appendix B-17.



When I used $a = 1$, the graph of $y = x^2$ appeared. The parabola was too narrow. It had to be vertically compressed to fit the arch. To do this, I needed a lower value of a, between 0 and 1. I needed a positive value because the arch opens upward.



I tried $a = 0.5$, but the parabola was not wide enough.

I tried $a = 0.1$. This value gave me a better fit. I still wasn't satisfied, so I tried different values of a between 0 and 0.1. I found that $a = 0.04$ gave me a good fit.

An equation that models the bridge is $y = 0.04x^2$.

Vertically compressing the graph of $y = x^2$ by a factor of 0.04 creates a graph that fits the photograph.

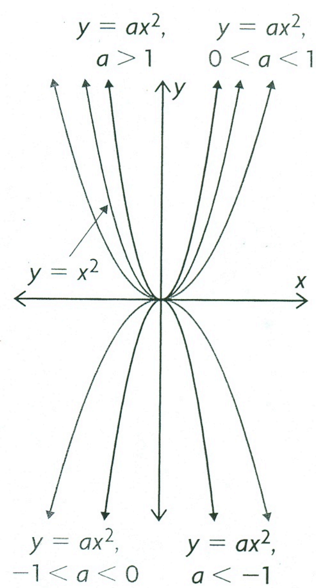
In Summary

Key Idea

- When compared with the graph of $y = x^2$, the graph of $y = ax^2$ is a parabola that has been stretched or compressed vertically by a factor of a.

Need to Know

- Vertical stretches are determined by the value of a. When $a > 1$, the graph is stretched vertically. When $a < -1$, the graph is stretched vertically and reflected across the x-axis.
- Vertical compressions are also determined by the value of a. When $0 < a < 1$, the graph is compressed vertically. When $-1 < a < 0$, the graph is compressed vertically and reflected across the x-axis.
- If $a > 0$, the parabola opens upward.
- If $a < 0$, the parabola opens downward.



CHECK Your Understanding

1. Match each graph with the correct equation. The graph of $y = x^2$ is in green in each diagram.

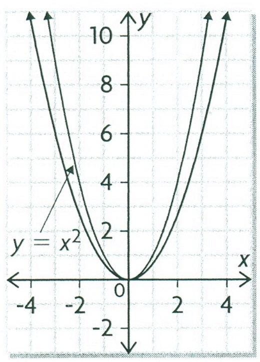
a) $y = 4x^2$

b) $y = -3x^2$

c) $y = \frac{2}{3}x^2$

d) $y = -0.4x^2$

i)



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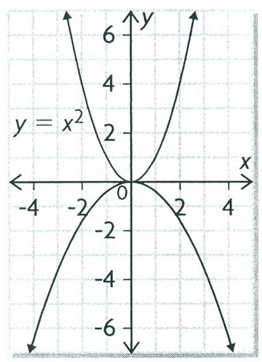
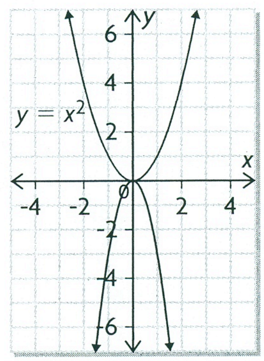


iii)



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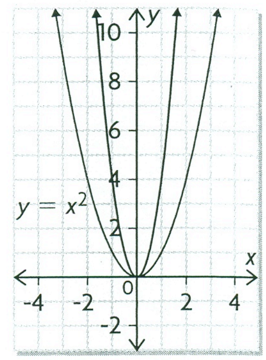


2. The graph of $y = x^2$ is transformed to $y = ax^2$ $(a \neq 1)$. For each point on $y = x^2$, determine the coordinates of the transformed point for the indicated value of a.

a) $(1, 1)$, when $a = 5$

b) $(-2, 4)$, when $a = -3$

c) $(5, 25)$, when $a = -0.6$

d) $(-4, 16)$, when $a = \frac{1}{2}$

3. Write the equations of two different quadratic relations that match each description.

a) The graph is narrower than the graph of $y = x^2$ near its vertex.

b) The graph is wider than the graph of $y = -x^2$ near its vertex.

c) The graph opens downward and is narrower than the graph of $y = 3x^2$ near its vertex.

PRACTISING

4. K Sketch the graph of each equation by applying a transformation to the graph of $y = x^2$. Use a separate grid for each equation, and start by sketching the graph of $y = x^2$.

a) $y = 3x^2$

b) $y = -0.5x^2$

c) $y = -2x^2$

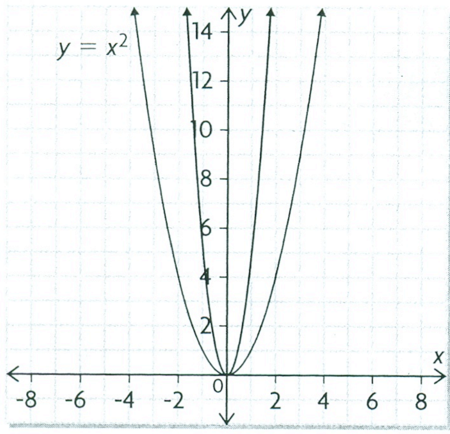
d) $y = \frac{1}{4}x^2$

e) $y = -\frac{3}{2}x^2$

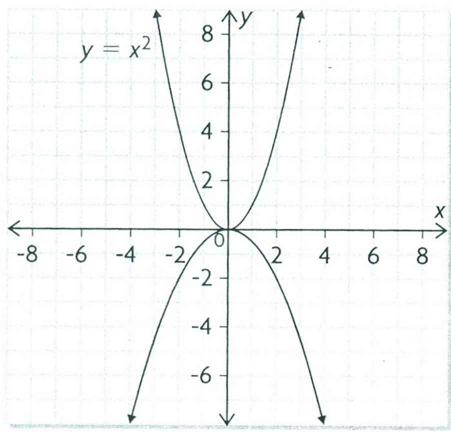
f) $y = 5x^2$

5. Describe the transformation(s) that were applied to the graph of $y = x^2$ to obtain each black graph. Write the equation of the **black** graph.

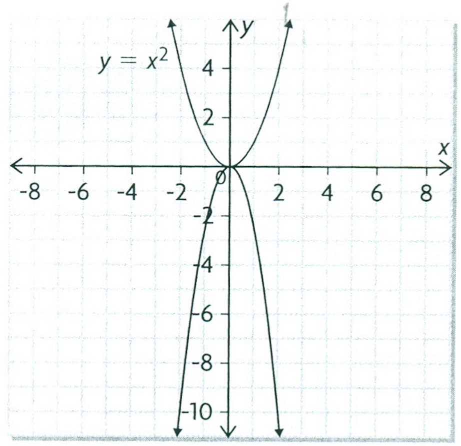
a)



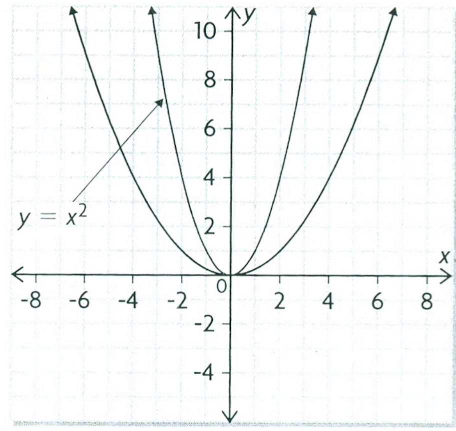
b)



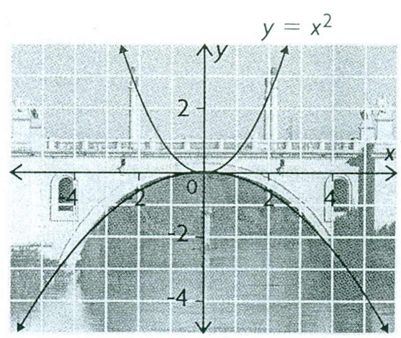
c)



d)

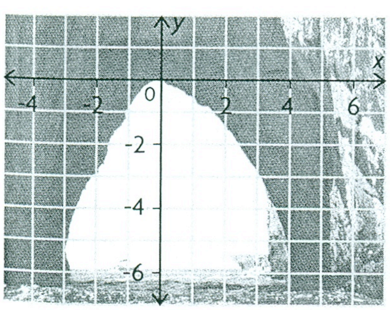


6. Andy modelled the arch of the bridge in the photograph at the right by tracing a parabola onto a grid. Now he wants to determine an equation of the parabola. Explain the steps he should use to do this, and state the equation.

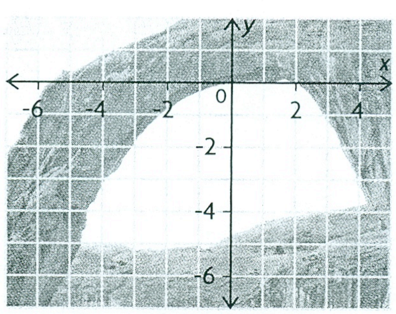


7. Determine an equation of a quadratic model for each natural arch.

a) Isle of Capri in Italy

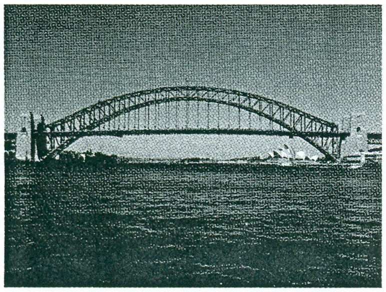


b) Corona Arch in Utah



8. Identify the transformation(s) that must be applied to the graph of $y = x^2$ to create a graph of each equation. Then state the coordinates of the image of the point (2, 4).

a) $y = 4x^2$
b) $y = -\frac{2}{3}x^2$
c) $y = 0.25x^2$
d) $y = -5x^2$
e) $y = -x^2$
f) $y = \frac{1}{5}x^2$



9. **A** By tracing the bridge at the left onto a grid, determine an equation that models the lower outline of the Sydney Harbour Bridge in Australia.

10. **T** Seth claims that changing the value of a in quadratic relations of the form $y = ax^2$ will never result in a parabola that is congruent to the parabola $y = x^2$. Do you agree or disagree? Justify your decision.

11. Copy and complete the following table.

Equation	Direction of Opening (upward/ downward)	Description of Transformation (stretch/ compress)	Shape of Graph Compared with Graph of $y = x^2$ (wider/narrower)
$y = 5x^2$			
$y = 0.25x^2$			
$y = -\frac{1}{3}x^2$			
$y = -8x^2$			

12. Explain why it makes sense that each statement about the graph of $y = ax^2$ is true.

a) If $a < 0$, then the parabola opens downward.
b) If a is a rational number between -1 and 1, then the parabola is wider than the graph of $y = x^2$.
c) The vertex is always (0, 0).

Extending

13. The graph of $y = ax^2$ ($a \neq 1$, $a > 0$) is either a vertical stretch or a vertical compression of the graph of $y = x^2$. Use graphing technology to determine whether changing the value of a has a similar effect on the graphs of equations such as $y = ax$, $y = ax^3$, $y = ax^4$, and $y = ax^{\frac{1}{2}}$.

14. The equation of a circle with radius r and centre (0, 0) is $x^2 + y^2 = r^2$.

a) Explore the effect of changing positive values of a when graphing $ax^2 + ay^2 = r^2$.
b) Explore the effects of changing positive values of a and b when graphing $ax^2 + by^2 = r^2$.

11. Answers may vary, e.g.,

Definition:	Special Properties:
A relation that can be described by an equation with a polynomial whose highest degree is 2	The graph has a vertical line of symmetry. The graph also has a single minimum or maximum value.
Quadratic Relation	
Examples: $y = x^2 + 9x + 2$; $y = 2(x + 4)(x - 6)$; $y = 4(x + 2)^2 - 3$	Non-examples: $y = x + 9$; $y = x^3 + 9x + 3$; $y = \sqrt{x}$

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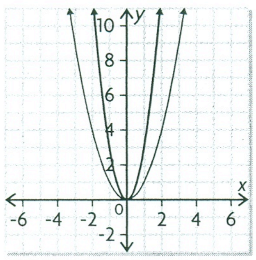
1. a) iv b) iii c) i d) ii

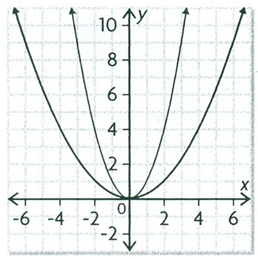
2. a) (1, 5) b) (−2, −12) c) (5, −15) d) (−4, 8)

3. a) Answers may vary, e.g., $y = 4x^2$; $y = 1.01x^2$
b) Answers may vary, e.g., $y = 0.5x^2$; $y = -0.1x^2$
c) Answers may vary, e.g., $y = -3.1x^2$; $y = -6x^2$

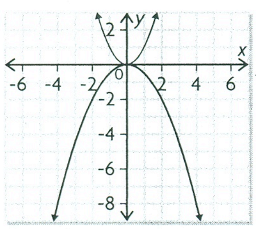
4. a)



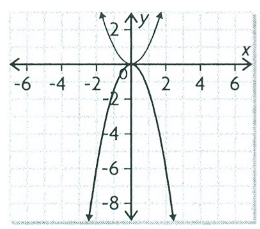
d)



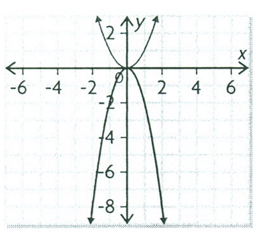
b)



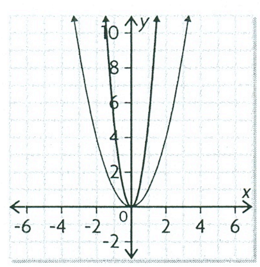
e)



c)



f)



5. a) vertical stretch by a factor of 4; $y = 4x^2$
b) vertical compression by a factor of $\frac{1}{2}$, reflected in the x-axis; $y = -\frac{1}{2}x^2$
c) vertical stretch by a factor of 2.5, reflected in the x-axis; $y = -2.5x^2$
d) vertical compression by a factor of $\frac{1}{4}$; $y = \frac{1}{4}x^2$

6. Choose the point (2, −0.5), and substitute this point into $y = ax^2$; solve for a; Answers may vary, e.g., $y = -0.125x^2$.

7. a) Answers may vary, e.g., $y = -\frac{5}{9}x^2$
b) Answers may vary, e.g., $y = -\frac{3}{16}x^2$

8. a) vertical stretch by a factor of 4; (2, 16)
b) reflection in the x-axis, vertical compression by a factor of $\frac{2}{3}$; $\left(2, -\frac{8}{3}\right)$
c) vertical compression by a factor of 0.25; (2, 1)
d) reflection in the x-axis, vertical stretch by a factor of 5; (2, −20)
e) reflection in the x-axis; (2, −4)
f) vertical compression by a factor of $\frac{1}{5}$; $\left(2, \frac{4}{5}\right)$

9. Answers may vary, e.g., $y = -\frac{1}{9}x^2$

10. Disagree. Changing the value of a to 1 or −1 will make $y = ax^2$ congruent to $y = x^2$.

11.

Equation	Direction of Opening (upward/ downward)	Description of Transformation (stretch/compress)	Shape of Graph Compared with Graph of $y = x^2$ (wider/narrower)
$y = 5x^2$	upward	stretch	narrower
$y = 0.25x^2$	upward	compress	wider
$y = -\frac{1}{3}x^2$	downward	compress	wider
$y = -8x^2$	downward	stretch	narrower

12. a) All the y-coordinates are multiplied by a negative number. This means that all the points on the graph $y = ax^2$ are reflected in the x-axis, causing the parabola to open downward.
b) The y-coordinates of the points on the graph are multiplied by a fraction whose magnitude is less than 1, so the points are moved toward the x-axis, making the parabola wider.
c) Since the y-coordinate of the vertex is 0, and multiplying 0 by any number results in a value of 0, the vertex is not affected.

13. It has the same effect on all graphs.

14. a) As the value of a increases, the radius of the circle decreases. As the value of a decreases, the radius of the circle increases.
b) The graph of $ax^2 + by^2 = r^2$ is a circle that has been stretched or compressed both horizontally and vertically for all values of a and b, where $a \neq 1$ and $b \neq 1$. The resulting oval shape is called an ellipse. As the value of a increases, the width of the oval shape along the x-axis decreases. As the value of a decreases, the width of the oval shape along the x-axis increases. As the value of b increases, the width of the oval shape along the y-axis decreases. As the value of b decreases, the width of the oval shape along the y-axis increases.

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1. a) $h = 3, k = 0; y = (x - 3)^2$
b) $h = 0, k = -4; y = x^2 - 4$
c) $h = -2, k = 0; y = (x + 2)^2$
d) $h = 0, k = 5; y = x^2 + 5$
e) $h = -6, k = -7; y = (x + 6)^2 - 7$
f) $h = 2, k = 5; y = (x - 2)^2 + 5$

2. a) iii b) v c) ii d) iv