2.2 Length of a Line Segment

GOAL

Determine the length of a line segment.

YOU WILL NEED

- grid paper
- ruler

INVESTIGATE the Math

Some computers can translate a handwritten entry into text by calculating the lengths of small line segments within the entry and comparing these lengths to stored information about the lengths of pieces of letters.

How can you use the coordinates of the endpoints of a line segment to determine its length?

A. Plot two points, A and B, on a grid so that line segment AB is neither horizontal nor vertical. Join A and B. Then construct a right triangle that has AB as its hypotenuse.

B. Write the coordinates of the vertex of the right angle. How are these coordinates related to the coordinates of the endpoints of the right angle?

C. Determine the lengths of the horizontal and vertical sides of the right triangle. How are these lengths related to the coordinates of A and B?

D. Calculate the length of AB.

E. Repeat parts A to D for line segment PQ, with endpoints $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Reflecting

F. Does it matter which point is (x_1, y_1), and which is (x_2, y_2), when calculating the length of a line segment? Explain.

G. Describe how to use each of the four coordinates of points P and Q to determine the length of PQ.

H. Why do you think the equation for calculating the length of a line segment is sometimes called the distance formula?

APPLY the Math

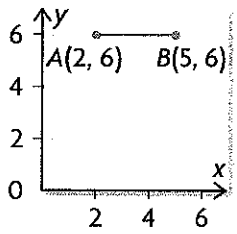
EXAMPLE 1 Selecting a strategy to calculate the length of a line segment

Determine the length of the line segment with each pair of endpoints.

a) $A(2, 6)$ and $B(5, 6)$ **b)** $G(-7, 8)$ and $H(-7, -5)$ **c)** $P(-4, 7)$ and $Q(3, 1)$

Niranjan's Solution

a)



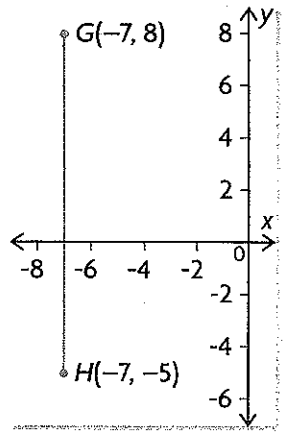
I noticed that A and B have the same y-coordinate, so I knew that AB was horizontal. I made a sketch to check.

$AB = 5 - 2$
$= 3$

I calculated the difference in the x-coordinates to determine the length of AB.

The length of AB is 3 units.

b)



I noticed that G and H have the same x-coordinate, so I knew that GH was vertical. I made a sketch to check.

$GH = 8 - (-5)$
$= 13$

I calculated the difference in the y-coordinates to determine the length of GH.

The length of GH is 13 units.

c) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

I noticed that the x- and y-coordinates of the endpoints are different numbers. I knew the line segment couldn't be horizontal or vertical. So, I used the distance formula.

$PQ = \sqrt{[3 - (-4)]^2 + (1 - 7)^2}$
$= \sqrt{7^2 + (-6)^2}$
$= \sqrt{49 + 36}$
$= \sqrt{85}$
$\doteq 9.2$

I chose $P(-4, 7)$ to be (x_1, y_1) and $Q(3, 1)$ to be (x_2, y_2). I substituted these values into the distance formula.

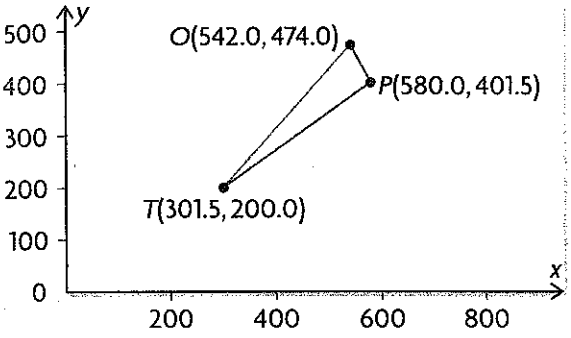
The length of PQ is approximately 9.2 units.

I rounded my answer to the nearest tenth of a unit.

EXAMPLE 2 Representing distances on a coordinate grid

Winston takes different routes to drive from his home in Toronto to Carleton University in Ottawa. Sometimes he takes Highway 401 to Prescott, and then Highway 416 to Ottawa. Other times he drives directly from Toronto to Ottawa along Highway 7. On a map of southeastern Ontario, with the origin at Windsor and the coordinates in kilometres, Toronto is at $T(301.5, 200.0)$, Prescott is at $P(580.0, 401.5)$, and Ottawa is at $O(542.0, 474.0)$. Approximately how far does Winston drive using each route?

Maria's Solution



I made a sketch on grid paper. I saw that I had to calculate the lengths of *TO*, *TP*, and *PO*, and then compare *TO* with *TP* + *PO*.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$TO = \sqrt{(542.0 - 301.5)^2 + (474.0 - 200.0)^2}$$
$$= \sqrt{240.5^2 + 274.0^2}$$
$$= \sqrt{57\,840.25 + 75\,076.00}$$
$$= \sqrt{132\,916.25}$$
$$\doteq 365$$

I used the distance formula to determine the length of each line segment.

I rounded my first answer to the nearest kilometre to determine the distance that Winston drives using the direct route.

$$TP = \sqrt{(580.0 - 301.5)^2 + (401.5 - 200.0)^2}$$
$$= \sqrt{278.5^2 + 201.5^2}$$
$$= \sqrt{77\,562.25 + 40\,602.25}$$
$$= \sqrt{118\,164.5}$$
$$\doteq 343.8$$

I knew that I was going to add the lengths *TP* and *PO*, so I rounded to the nearest tenth of a kilometre because I planned to use these distances in another calculation.

$$PO = \sqrt{(542.0 - 580.0)^2 + (474.0 - 401.5)^2}$$
$$= \sqrt{(-38.0)^2 + 72.5^2}$$
$$= \sqrt{1444.00 + 5256.25}$$
$$= \sqrt{6700.25}$$
$$\doteq 81.9$$

$TP + PO = 343.8 + 81.9$
$\doteq 426$

I added lengths *TP* and *PO* to determine the distance that Winston drives using the indirect route. I rounded the distance to the nearest kilometre.

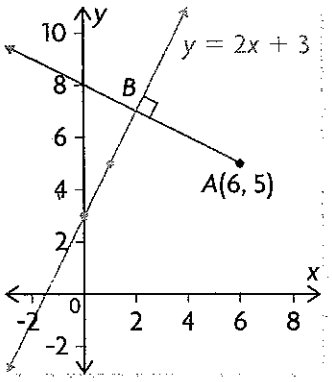
The route from Toronto directly to Ottawa is approximately 365 km. The route through Prescott is approximately 426 km.

These distances are estimates because they don't take into account turns in the road. Even though the route along Highways 401 and 416 is longer, it might be faster since Winston can travel at a greater speed on multi-lane highways.

EXAMPLE 3 Reasoning to determine the distance between a point and a line

Calculate the distance between point $A(6, 5)$ and the line $y = 2x + 3$.

Kerry's Solution



I graphed the line by plotting the *y*-intercept at (0, 3). Then I used the slope to determine a second point on the line. I drew a straight line between these points. I also plotted point *A*.

I reasoned that the distance I needed to calculate was the shortest distance between point *A* and the line. I figured that this distance would be measured on a line through *A*, perpendicular to $y = 2x + 3$.

I called the point where the red line and the blue line intersect point *B*.

I had to calculate the length of *AB*. To do this, I needed the coordinates of *B*, which meant that I needed the equation of the red line.

The slope of $y = 2x + 3$ is 2.
The slope of *AB* is $-\frac{1}{2}$.

Since the red line is perpendicular to the blue line, the slope of *AB* is the negative reciprocal of 2.

Therefore, $y = -\frac{1}{2}x + b$ is an equation for the red line.

$5 = -\frac{1}{2}(6) + b$

I substituted the coordinates of *A* into the equation to determine *b*.

$5 = -3 + b$
$8 = b$

Therefore, $y = -\frac{1}{2}x + 8$ is the equation of the red line.

$y = 2x + 3$
$y = -\frac{1}{2}x + 8$

To determine the point of intersection, I had to solve a system of equations.

$-\frac{1}{2}x + 8 = 2x + 3$

I used substitution to replace y in the first equation with the right side of the second equation. Then I solved for x.

$2\left(-\frac{1}{2}\right)x + 2(8) = 2(2x) + 2(3)$
$-x + 16 = 4x + 6$
$16 - 6 = 4x + x$
$10 = 5x$
$2 = x$

$y = 2(2) + 3$
$y = 7$

I let $x = 2$ in the first equation to determine the value of y.

The coordinates of B are $(2, 7)$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$d_{AB} = \sqrt{(2 - 6)^2 + (7 - 5)^2}$
$= \sqrt{(-4)^2 + 2^2}$
$= \sqrt{20}$
$\doteq 4.5$

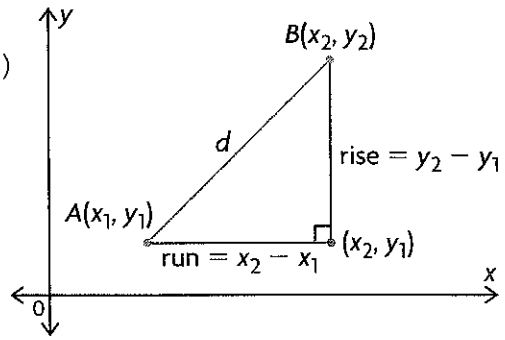
I used the distance formula to calculate the length of AB, where $A(x_1, y_1) = A(6, 5)$ and $B(x_2, y_2) = B(2, 7)$.

The point $(6, 5)$ is about 4.5 units away from the line $y = 2x + 3$.

In Summary

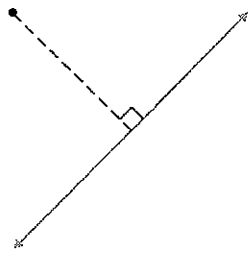
Key Idea

- The distance, d, between the endpoints of a line segment, $A(x_1, y_1)$ and $B(x_2, y_2)$, can be calculated using the distance formula:
  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Need to Know

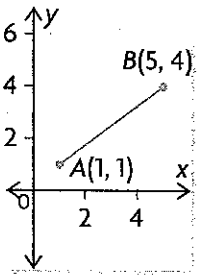
- The Pythagorean theorem is used to develop the distance formula, by calculating the straight-line distance between two points.
- The distance between a point and a line is the shortest distance between them. It is measured on a perpendicular line from the point to the line.



CHECK Your Understanding

1. Determine the length of each line segment.

a)



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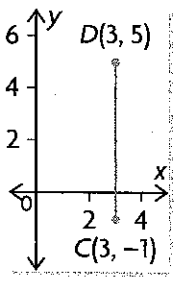
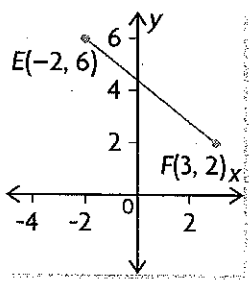
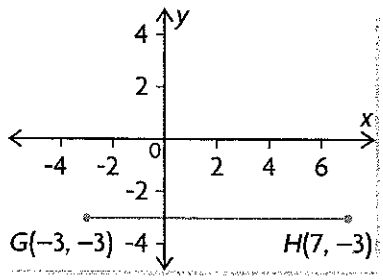


c)



d)



2. For each pair of points:

i) Draw the line segment joining the points.

ii) Determine the length of the line segment.

a) $P(-4, 4)$ and $Q(3, 1)$

b) $R(2, -1)$ and $S(10, 2)$

c) $T(3.5, -3)$ and $U(3.5, 11)$

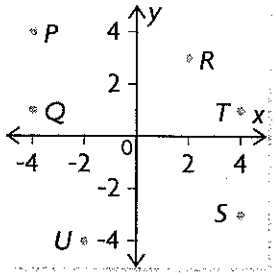
d) $X(-1, 6)$ and $Y(5, 6)$

3. A helicopter travelled from Kapuskasing to North Bay. On a map of Ontario, with the origin at Windsor and the coordinates in kilometres, Kapuskasing is at $K(-70, 770)$ and North Bay is at $N(220, 490)$.

a) Approximately how far did the helicopter travel?

b) What assumption did you make about the route of the helicopter?

PRACTISING



4. Calculate the distance between each pair of points in the diagram at the left.

a) P and Q

b) Q and R

c) U and S

d) P and R

e) P and U

f) Q and T

5. For each pair of points below:

i) Draw the line segment joining the points.

ii) Calculate the length of the line segment.

a) $A(2, 6)$ and $B(5, 2)$

b) $C(-3, 4)$ and $D(3, 2)$

c) $E(-6, 8)$ and $F(-6, -9)$

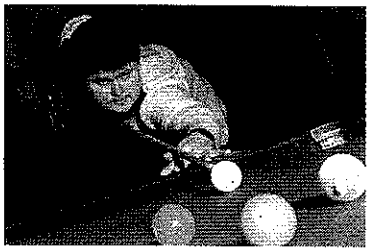
d) $G(0, -7)$ and $H(1, 3)$

e) $I(-3, -3)$ and $J(5, -4)$

f) $K(-10, -2)$ and $L(6, -2)$

6. a) Which line segment(s) for question 5 are vertical? Which are horizontal? Explain how you know.

b) How can you calculate the length of a vertical or horizontal line segment without using the distance formula?



7. A coordinate system is superimposed on a billiard table. Gord has a yellow ball at $A(2, 3)$. He is going to "bank" it off the side rail at $B(6, 5)$, into the pocket at $C(2, 7)$. How far will the yellow ball travel?

8. Which of these points is closest to point $A(-3.2, 5.6)$: $B(1.8, -4.3)$, K $C(0.7, 8.9)$, or $D(-7.6, 3.9)$? Justify your decision.

9. A forest fire is threatening two small towns, Mordon and Bently. On a map, the fire is located at (10, −11), the fire hall in Mordon is located at (26, 77), and the fire hall in Bently is located at (12, −88). Which fire hall is closer to the fire?

10. In a video game, three animated characters are programmed to run out of a building at $F(1, -1)$ and head in three different directions. After 2 s, Animal is at $A(22, 18)$, Beast is at $B(-3, 35)$, and Creature is at $C(7, -29)$. Which character ran farthest?

11. C How are the formulas for calculating the length of a line segment and the midpoint of a line segment, using the coordinates of the endpoints, the same? How are they different?

12. Calculate the distance between each line and the point. Round your answer to one decimal place.

a) $y = 4x - 2, (-3, 3)$ **c)** $2x + 3y = 6, (7, 6)$

b) $y = -x + 5, (-1, -2)$ **d)** $5x - 2y = 10, (2, 4.5)$

13. I A new amusement park is going to be built near two major highways. On a coordinate grid of the area, with the scale 1 unit represents 1 km, the park is located at $P(3, 4)$. Highway 2 is represented by the equation $y = 2x + 5$, and Highway 10 is represented by the equation $y = -0.5x + 2$. Determine the coordinates of the exits that must be built on each highway to result in the shortest road to the park.

14. A coordinate grid is superimposed on the plan of a new housing development. A fibre-optic cable is being laid to link points $A(-18, 12)$, $B(-8, 1)$, $C(3, 4)$, and $D(15, 7)$ in a run beginning at A and ending at D. If one unit on the grid represents 2.5 m, how much cable is required?

15. A leash-free area for dogs is going to be created in a field behind a recreation centre. The area will be in the shape of an irregular pentagon, with vertices at (2, 0), (1, 6), (8, 9), (10, 7), and (6, 0). If one unit on the plan represents 10 m, what length of fencing will be required?

16. Suppose that you know the coordinates of three points. Explain how you would determine which of the first two points is closer to the third point. Describe the procedures, facts, and formulas you would use, and give an example.

Extending

17. $\triangle ABC$ has vertices at $A(1, 2)$, $B(4, 8)$, and $C(8, 4)$.

a) $\triangle ABC$ is translated so that vertex A' is on the x-axis and vertex B' is on the y-axis. Determine the coordinates of the translated triangle, $\triangle A'B'C'$.

b) $\triangle DEF$ has vertices at $D(-1, 1)$, $E(-2, 6)$, and $F(-8, 3)$. Is $\triangle DEF$ congruent to $\triangle ABC$? Justify your answer.