1.6 Solving Linear Systems: Elimination

GOAL

Solve a linear system of equations using equivalent equations to remove a variable.

LEARN ABOUT the Math

Every day, Brenna bakes chocolate chip and low-fat oatmeal cookies in her bakery. She uses different amounts of butter and oatmeal in each recipe. Brenna has 47 kg of butter and 140 kg of oatmeal.

Chocolate Chip	Low-Fat Oatmeal
• 13 kg butter	• 2 kg butter
• 8 kg oatmeal	• 29 kg oatmeal

? How many batches of chocolate chip and low-fat oatmeal cookies can Brenna bake?

EXAMPLE 1 **Selecting an algebraic strategy** to eliminate a variable

Determine the number of batches of each type of cookie that Brenna can bake using all the butter and oatmeal she has.

Chantal's Solution: Selecting an algebraic strategy to eliminate a variable

Let r represent the number of batches of chocolate chip cookies. Let s represent the number of batches of low-fat cookies.

I used variables for the numbers of batches.

$13r + 2s = 47$ ① butter

$8r + 29s = 140$ ② oatmeal

I wrote two equations, ① to represent the amount of butter and ② the amount of oatmeal. I decided to use an **elimination strategy** to eliminate the r terms by subtracting two equations.

elimination strategy

a method of removing a variable from a system of linear equations by creating an equivalent system in which the coefficients of one of the variables are the same or opposites

$8(13r + 2s) = 8(47)$ ① × 8

$8(13r) + 8(2s) = 8(47)$

$104r + 16s = 376$

$13(8r + 29s) = 13(140)$ ② × 13

$13(8r) + 13(29s) = 13(140)$

$104r + 377s = 1820$

To eliminate the r terms by subtracting, I had to make the coefficients of the r terms the same in both equations. I multiplied equation ① by 8 and equation ② by 13.

Communication | *Tip*

The steps that are required to eliminate a variable can be described by showing the operation and the equation number. For example, "① × 8" means "equation ① multiplied by 8."

$$\begin{aligned} 104r + 16s &= 376 \\ 104r + 377s &= 1820 \\ \hline -361s &= -1444 \end{aligned}$$

I subtracted the equations to eliminate *r*. ① × 8 − ② × 13

$$s = \frac{-1444}{-361}$$
$$s = 4$$

I solved for *s*.

$$\begin{aligned} 13r + 2(4) &= 47 \\ 13r + 8 &= 47 \\ 13r &= 47 - 8 \\ 13r &= 39 \\ r &= \frac{39}{13} \\ r &= 3 \end{aligned}$$

I substituted the value of *s* into equation ①. (I could have used equation ② instead, if I had wanted.) I solved for *r*.

Brenna can make three batches of chocolate chip cookies and four batches of low-fat cookies.

Check:

I verified my answers.

Type of Cookie	**Number of Batches**	**Butter (kg)**	**Oatmeal (kg)**
chocolate chip	3	3 × 13 = 39	3 × 8 = 24
low-fat	4	4 × 2 = 8	4 × 29 = 116
Total		39 + 8 = 47	24 + 116 = 140

Leif's Solution: Selecting an algebraic strategy to eliminate a different variable

$$\begin{aligned} 13r + 2s &= 47 \quad ① \\ 8r + 29s &= 140 \quad ② \end{aligned}$$

I started with the same linear system as Chantal, but I decided to eliminate the *s* terms by adding the two equations.

$$\begin{aligned} 29(13r + 2s) &= 29(47) \quad ① \times 29 \\ 29(13r) + 29(2s) &= 29(47) \\ 377r + 58s &= 1363 \\ -2(8r + 29s) &= -2(140) \quad ② \times -2 \\ -2(8r) - 2(29s) &= -2(140) \\ -16r - 58s &= -280 \end{aligned}$$

To eliminate the *s* terms by adding, I had to make the coefficients of the *s* terms opposites. To do this, I multiplied equation ① by 29 and equation ② by −2.

$$\begin{aligned} 377r + 58s &= 1363 \\ -16r - 58s &= -280 \\ \hline 361r \quad &= 1083 \end{aligned}$$

I added the equations to eliminate s. ① × 29 − ② × −2

$$r = \frac{1083}{361}$$
$$r = 3$$

I solved for r.

$$\begin{aligned} 13(3) + 2s &= 47 \\ 39 + 2s &= 47 \\ 2s &= 47 - 39 \\ 2s &= 8 \\ s &= \frac{8}{2} \\ s &= 4 \end{aligned}$$

I substituted the value of r into equation ①.

Brenna can make three batches of chocolate chip cookies and four batches of low-fat cookies.

Reflecting

A. How did Chantal and Leif use elimination strategies to change a system of two equations into a single equation?

B. Why did Chantal and Leif need to multiply both equations to eliminate a variable?

C. Explain when you would add and when you would subtract to eliminate a variable.

D. Whose strategy would you choose: Chantal's or Leif's? Why?

APPLY the Math

EXAMPLE 2 Selecting an elimination strategy to solve a linear system

Use elimination to solve this linear system:

$7x - 12y = 42$

$17x + 8y = -2$

John's Solution

$$7x - 12y = 42 \quad ①$$
$$17x + 8y = -2 \quad ②$$

I decided to eliminate the y terms because I prefer to add. Since their signs were different, I could make the coefficients of the y terms opposites.

$$\begin{aligned} 14x - 24y &= 84 \quad ① \times 2 \\ 51x + 24y &= -6 \quad ② \times 3 \\ \hline 65x &= 78 \\ x &= \frac{78}{65} \\ x &= 1.2 \end{aligned}$$

The coefficients of y are factors of 24. I multiplied equation ① by 2 and equation ② by 3 to make the coefficients of the y terms opposites. Then I added the new equations to eliminate y. ① × 2 + ② × 3

$$\begin{aligned} 7(1.2) - 12y &= 42 \quad ① \\ 8.4 - 12y &= 42 \\ -12y &= 42 - 8.4 \\ -12y &= 33.6 \\ y &= \frac{33.6}{-12} \\ y &= -2.8 \end{aligned}$$

I substituted the value of x into equation ① and solved for y.

Verify by substituting $x = 1.2$ and $y = -2.8$ into both original equations.

$$7x - 12y = 42$$

Left Side	Right Side
$7x - 12y$	42
$= 7(1.2) - 12(-2.8)$	
$= 8.4 + 33.6$	
$= 42$	

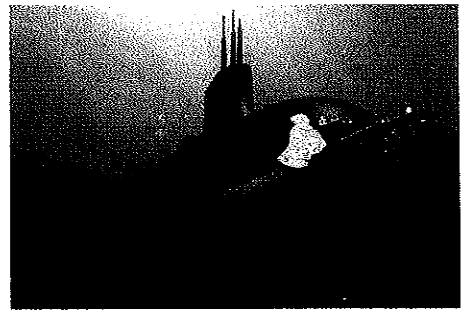
$$17x + 8y = -2$$

Left Side	Right Side
$17x + 8y$	-2
$= 17(1.2) + 8(-2.8)$	
$= 20.4 - 22.4$	
$= -2$	

The solution is $(1.2, -2.8)$.

EXAMPLE 3 **Selecting an elimination strategy** to solve a system with rational coefficients

During a training exercise, a submarine travelled 20 km/h on the surface and 10 km/h underwater. The submarine travelled 200 km in 12 h. How far did the submarine travel underwater?



Tanner's Solution

Let x represent the distance that the submarine travelled on the surface. Let y represent the distance that it travelled underwater.

I used variables for the distances that the submarine travelled.

$x + y = 200$ ①

I wrote an equation for the total distance travelled during the training exercise.

The time spent on the surface is $\frac{x}{20}$.

The time spent underwater is $\frac{y}{10}$.

$\frac{x}{20} + \frac{y}{10} = 12$ ②

I used the formula time $= \frac{\text{distance}}{\text{speed}}$ to write expressions for the time spent on the surface and the time spent underwater. Then I wrote an equation for the total time.

$$20\left(\frac{x}{20} + \frac{y}{10}\right) = 20(12) \quad ② \times 20$$
$$20\left(\frac{x}{20}\right) + 20\left(\frac{y}{10}\right) = 20(12)$$
$$x + 2y = 240 \quad ② \times 20$$

I created an equivalent system with no fractional coefficients by multiplying equation ② by 20, since 20 is a common multiple of 20 and 10.

$$\begin{aligned} x + y &= 200 \quad ① \\ x + 2y &= 240 \quad ② \times 20 \\ \hline -y &= -40 \\ y &= 40 \end{aligned}$$

Since the coefficients of x were now the same, I decided to eliminate x by subtracting the equations.

$$\begin{aligned} x + 40 &= 200 \\ x &= 200 - 40 \\ x &= 160 \end{aligned}$$

To determine x, I substituted 40 for y in the equation $x + y = 200$.

The submarine travelled 40 km underwater.

In Summary

Key Idea

- To eliminate a variable from a system of linear equations, you can
  - add two equations when the coefficients of the variable are opposite integers
  - subtract two equations when the coefficients of the variable are the same

Need to Know

- Elimination is a convenient strategy when the variable you want to eliminate is on the same side in both equations.
- If there are fractional coefficients in a system of equations, you can form equivalent equations without fractional coefficients by choosing a multiplier that is a common multiple of the denominators.
- Adding, subtracting, multiplying, or dividing both sides of a linear equation in the same way produces an equation that is equivalent to the original equation.

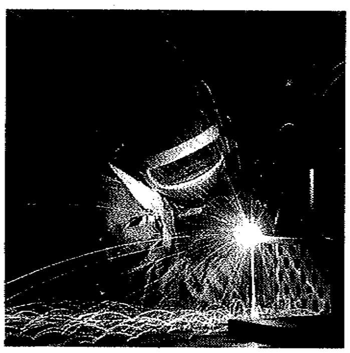
CHECK Your Understanding

1. For each linear system, state whether you would add or subtract to eliminate one of the variables without using multiplication.

a) $4x + y = 5$, $3x + y = 7$ **b)** $3x - 2y = 8$, $5x - 2y = 9$ **c)** $4x - 3y = 6$, $4x + 7y = 9$ **d)** $4x - 5y = 4$, $3x + 5y = 10$

2. **a)** Describe how you would eliminate the variable x from the system of equations in question 1, part a).
   b) Describe how you would eliminate the variable x from the system of equations in question 1, part c).

3. When a welder works for 3 h and an apprentice works for 5 h, they earn a total of \$175. When the welder works for 7 h and the apprentice works for 8 h, they earn a total of \$346. Determine the hourly rate for each worker.



Safety *Connection*

Welders must wear a helmet with a mask that has a darkened lens, as well as gloves and clothing that are flame- and heat-resistant.

PRACTISING

4. To eliminate y from each linear system, by what numbers would you multiply equations ① and ②?

a) $4x + 2y = 5$ ① , $3x - 4y = 7$ ②

b) $3x - 7y = 11$ ① , $5x + 8y = 9$ ②

c) $4x + 3y = 12$ ① , $-2x + 5y = 7$ ②

d) $9x - 4y = 10$ ① , $3x + 2y = 10$ ②

5. To eliminate x from each linear system in question 4, by what numbers would you multiply equations ① and ②?

6. Solve each system by using elimination.

K **a)** $3x + y = -2$
$x - y = -6$

b) $x + 5y = 1$
$2x + 3y = 9$

c) $4x - y = 5$
$-5x + 2y = -1$

d) $2x - 3y = -2$
$3x - y = 0.5$

e) $3x - 2y = -39$
$x + 3y = 31$

f) $5x - y = -3.8$
$4x + 3y = 7.6$

7. Determine, without graphing, the point of intersection for the lines with equations $x + 3y = -1$ and $4x - y = 22$. Verify your solution.

8. In a charity walkathon, Lori and Nicholas walked 72.7 km. Lori walked 8.9 km farther than Nicholas.

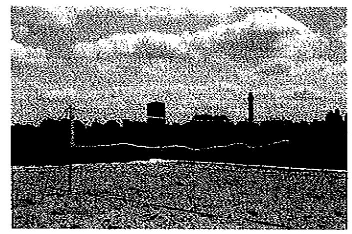
a) Create a linear system to model this situation.

b) Solve the system to determine how far each person walked.

9. The perimeter of a beach volleyball court is 54 m. The difference between its length and its width is 9 m.

a) Create a linear system to model this situation.

b) Solve the system to determine the dimensions of the court.



10. Rolf needs 500 g of chocolate that is 86% cocoa for a truffle recipe.
A He has one kind of chocolate that is 99% cocoa and another kind that is 70% cocoa. How much of each kind of chocolate does he need to make the 86% cocoa blend? Round your answer to the nearest gram.

11. Determine the point of intersection for each pair of lines. Verify your solution.

a) $4x + 7y = 23$
$6x - 5y = -12$

b) $\frac{x}{11} - \frac{y}{8} = -2$
$\frac{x}{2} - \frac{y}{4} = 3$

c) $0.5x - 0.3y = 1.5$
$0.2x - 0.1y = 0.7$

d) $\frac{x}{2} - 5y = 7$
$3x + \frac{y}{2} = \frac{23}{2}$

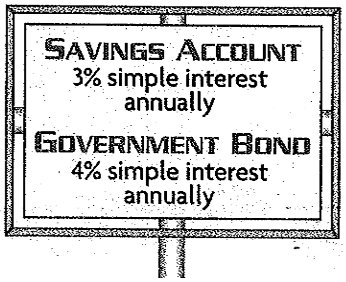
e) $5x - 12y = 1$
$13x + 9y = 16$

f) $\frac{x}{9} + \frac{y - 3}{3} = 1$
$\frac{x}{2} - (y + 9) = 0$

12. Each gram of a mandarin orange has 0.26 mg of vitamin C and 0.13 mg of vitamin A. Each gram of a tomato has 0.13 mg of vitamin C and 0.42 mg of vitamin A. How many grams of mandarin oranges and tomatoes have 13 mg of vitamin C and 20.7 mg of vitamin A?

13. On weekends, as part of his exercise routine, Carl goes for a run, partly
C on paved trails and partly across rough terrain. He runs at 10 km/h on the trails, but his speed is reduced to 5 km/h on the rough terrain. One day, he ran 12 km in 1.5 h. How far did he run on the rough terrain?

14. Two fractions have denominators 3 and 4. Their sum is $\frac{17}{12}$. If the numerators are switched, the sum is $\frac{3}{2}$. Determine the two fractions.



15. A student athletic council raised $6500 in a volleyball marathon. The students put some of the money in a savings account and the rest in a government bond. The rates are shown at the left. After one year, the students earned $235. How much did they invest at each rate?

16. The caterers for a Grade 10 semi-formal dinner and dance are preparing two different meals: chicken at $12 or pasta at $8. The total cost of the dinners for 240 students is $2100.
  a) How many chicken dinners did the students order?
  b) How many pasta dinners did they order?

17. A magic square is an array of numbers with the same sum across any row, column, or main diagonal.

16	2	B
A		14
8		

24	$\frac{A}{2}$	18
9		
B		

  a) Determine a system of linear equations you can use to determine the values of A and B in both squares.
  b) What are the values of A and B?

18. Explain what it means to eliminate a variable from a linear system. Use the linear system $3x + 7y = 31$ and $5x - 8y = 91$ to compare different strategies for eliminating a variable.

Extending

19. The sum of the squares of two negative numbers is 74. The difference of their squares is 24. Determine the two numbers.

20. Solve the system $2xy + 3 = 4y$ and $3xy + 2 = 5y$.

21. A general system of linear equations is

$$ax + by = e$$
$$cx + dy = f$$

where a, b, c, d, e, and f are constant values.
  a) Use elimination to solve for x and y in terms of a, b, c, d, e, and f.
  b) Are there any values that a, b, c, d, e, and f cannot have?