3.9 Solving Problems with Quadratic Equations

Part 1: Solving a Quadratic Equation Graphically

Trendy Fashion Shop sells blouses. Over the last season, the manager used the quadratic relation $R = 300 + 20x - x^2$ to model the effect on revenue of raising or lowering the price. Here, R is the revenue in dollars and x is the price change in dollars.

Think, Do, Discuss

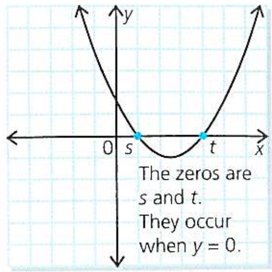
1. Graph the relation.
2. Use the graph to determine the price change that produces the maximum revenue.
3. (a) What is the price change that results in revenue of \$375?
   (b) What is the price change that results in revenue of \$300?
   (c) At what amount of increase or decrease does the shop get no revenue?
4. Each question in step 3 corresponds to an equation that must to be solved.
   (a) Write the equation that matches each question.
   (b) Why can't these equations be solved using the methods for solving linear equations that you learned in your previous mathematics courses?

Did You Know?

Are humans the only animals that count? No, say Nobuyuki Kawai and Tetsuro Matsuzawa of the Kyoto Research Institute of Japan. In April 2000, they reported in *Nature* that a 14-year-old chimpanzee named Ai has learned to use Arabic numerals to count from 0 to 9. Ai can also remember a five-digit sequence of numbers. Do some research. How do Ai's math skills compare to those of a preschool child? What other evidence is there that chimpanzees plan ahead?

Part 2: Solving a Quadratic Equation Algebraically

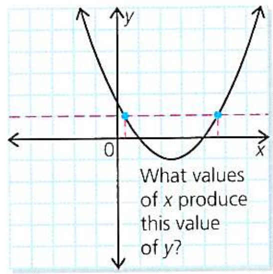
The earlier sections of this chapter focused on the zeros or x-intercepts of quadratic relations.



For a quadratic relation in standard form ($y = ax^2 + bx + c$) or in factored form ($y = a(x - s)(x - t)$) the zeros are found by solving for x when $y = 0$.

If $0 = a(x - s)(x - t)$, then $x = s$ and $x = t$ are the solutions of the equation.

In this section, you will learn to solve quadratic equations for *any* value of y, including zero.



Think, Do, Discuss

1. Consider the quadratic relation $y = x^2 - 8x + 12$.
   - **(a)** Graph the relation. Use factoring to determine the zeros exactly.
   - **(b)** Use the graph to find the value of x for which the value of the relation is -3. How accurate is your answer? How can you check?
   - **(c)** Write the equation that corresponds to the relation you found in (b).
   - **(d)** Rewrite the equation so that 0 is on one side and all the other terms on the other.
   - **(e)** Graph the relation that corresponds to this new equation. Use factoring to determine the zeros exactly.
   - **(f)** How does finding the zeros for this quadratic relation also answer the problem in (b)? Why is it possible to find an exact answer to the problem?
2. Why is solving $x^2 - 8x + 12 = -3$ the same as solving $x^2 - 8x + 15 = 0$?
3. Describe the key steps that you followed in question 1 to find the exact solution to an equation involving a quadratic expression.
4. Use this procedure to find exact values of x for which the relation has a value of 12, 5, and -4.
5. Try to use the procedure to find the exact value of x for which the relation has a value of 1. Why does the method not work in this case?
6. Under what circumstances will this procedure work to find the exact solution to an equation involving a quadratic expression?

Focus 3.9

Key Ideas

- If the value of y is known for the relation $y = ax^2 + bx + c$, then the corresponding values of x can be found either graphically or algebraically.
- If the quadratic relation $y = ax^2 + bx + c$ is graphed, then for any value of the dependent variable, y, the values of the independent variable, x, can be read from the graph.
- In a quadratic relation, when y is replaced with a number, the result is a **quadratic equation**. For example, $y = x^2 - 9$ is a quadratic relation and $72 = x^2 - 9$ is a quadratic equation.
- In some cases, quadratic equations can be solved by factoring. Follow this procedure:
  - Replace y with the given value.
  - Rearrange the quadratic equation to the form $ax^2 + bx + c = 0$.
  - Factor the quadratic expression, if possible.
  - Set each factor equal to zero and solve the resulting equations.

  For example, to solve $x^2 - x = 6$,

  $$x^2 - x = 6$$
  $$x^2 - x - 6 = 0$$
  $$(x + 2)(x - 3) = 0$$
  $$x + 2 = 0 \text{ or } x - 3 = 0$$
  $$x = -2 \text{ or } x = 3$$
- The solutions to a quadratic equation are often called the **roots** of the equation.

Example 1

A ball is thrown straight down from a 180 m high cliff.

The relation $h = -5t^2 - 5t + 180$ is a model that gives the approximate height of the ball h, in metres, at t seconds after it is thrown. How long does it take the ball to reach a ledge 80 m from the base of the cliff? Check the answer using a graphing calculator.

Solution

Since $h = -5t^2 - 5t + 180$, then for $h = 80$

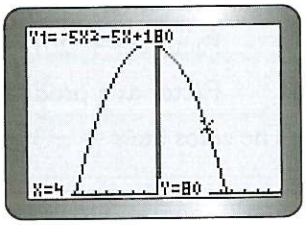
$80 = -5t^2 - 5t + 180$	
$0 = -5t^2 - 5t + 180 - 80$	Subtract 80 from both sides of the equation.
$0 = -5t^2 - 5t + 100$	Remove the common factor.
$0 = -5(t^2 + t - 20)$	Factor $t^2 + t - 20$.
$0 = -5(t + 5)(t - 4)$	Since $(5)(-4) = -20$ and $5 + (-4) = 1$.

So $t + 5 = 0$ or $t - 4 = 0$. The roots are $t = -5$ and $t = 4$.

The value $t = -5$ has no meaning in this context, since the ball only started moving at $t = 0$.

Therefore, after 4 s the ball reaches a height of 80 m from the base of the cliff.

Check with a graphing calculator. Graph the parabola and adjust the window. Use the **1:value** command from the CALCULATE menu ([2nd] [TRACE]), enter $x = 4$, and press [ENTER].



The result shows that the answer is correct.

Example 2

The population of a city is modelled by the relation $P = 0.5t^2 + 10t + 200$, where P is the population in thousands and t is the time in years. **Note:** $t = 0$ corresponds to the year 2000.

(a) What is the population in 2000?

(b) What is the population in 2002?

(c) When is the population 350 000? Explain your answer.

(d) Use a graphing calculator to check the answers.

Solution

Use a graphing calculator to check each of (a), (b), and (c) as they are completed.

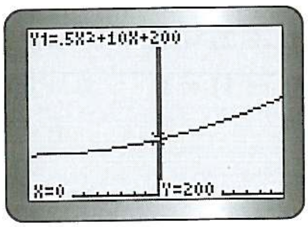
(a) When $t = 0$, the year is 2000. Substitute $t = 0$ in the equation.

$$P = 0.5t^2 + 10t + 200$$

$$P = 0.5(0)^2 + 10(0) + 200$$

$$P = 200$$

Since P is measured in thousands, the population in 2000 is 200 000. This screen capture shows the result is correct.



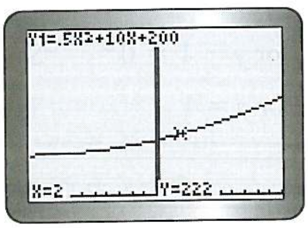
(b) The year 2002 corresponds to $t = 2$.
Substitute $t = 2$.

$$P = 0.5(2)^2 + 10(2) + 200$$

$$P = 0.5(4) + 20 + 200$$

$$P = 222$$

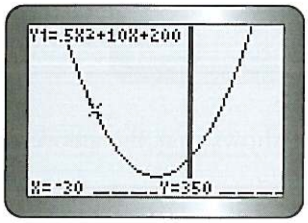
In 2002, the population is 222 000.



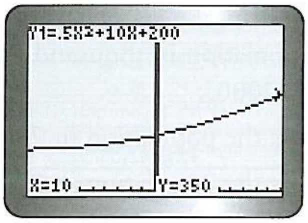
(c) To calculate when the population reaches 350 000, substitute $P = 350$ in the equation and solve for t.

$P = 0.5t^2 + 10t + 200$	
$350 = 0.5t^2 + 10t + 200$	Substitute $P = 350$.
$0 = 0.5t^2 + 10t + 200 - 350$	Subtract 350 from both sides of the equation.
$0 = 0.5t^2 + 10t - 150$	Remove the common factor.
$0 = 0.5(t^2 + 20t - 300)$	Factor $t^2 + 20t - 300$.
$0 = 0.5(t + 30)(t - 10)$	Factor as a product of binomials.

So $t + 30 = 0$ or $t - 10 = 0$. The zeros are $t = -30$ and $t = 10$ as shown.



The population will reach 350 000 in 2010. According to the model, the population was also 350 000 in 1970 (at $t = -30$). The relation models a city whose population decreased to a minimum and is now increasing.



Example 3

Solve the equation $18y - 14 = 4y^2$. Check your solution.

Solution

$4y^2 = 18y - 14$	
$4y^2 - 18y + 14 = 0$	Rewrite the equation in the form $ax^2 + bx + c = 0$.
$2(2y^2 - 9y + 7) = 0$	Remove the common factor.
$2(2y - 7)(y - 1) = 0$	Factor the trinomial as you have learned in this chapter. Check your factoring by expanding using the distributive property: $2y(y) + (-7)y + 2y(-1) + (-7)(-1)$ $= 2y^2 - 7y - 2y + 7$ $= 2y^2 - 9y + 7$
$2y - 7 = 0$ or $y - 1 = 0$	Set the factors equal to zero and solve.
$y = \frac{7}{2}$ or $y = 1$	The roots are $\frac{7}{2}$ and 1.

Check $y = \frac{7}{2}$:

L. S.	R. S.
$4y^2$	$18y - 14$
$= 4\left(\frac{7}{2}\right)^2$	$= 18\left(\frac{7}{2}\right) - 14$
$= 4\left(\frac{49}{4}\right)$	$= 9(7) - 14$
$= 49$	$= 63 - 14$
	$= 49$

left side = right side

Check $y = 1$:

L. S.	R. S.
$4y^2$	$18y - 14$
$= 4(1)^2$	$= 18(1) - 14$
$= 4$	$= 4$

left side = right side

The solutions to $18y - 14 = 4y^2$ are $y = \frac{7}{2}$ and $y = 1$.

Practise, Apply, Solve 3.9

A

1. Solve each equation.

(a) $(x + 5)(x - 2) = 0$
(b) $3y(y - 5) = 0$
(c) $(2m + 1)(m - 3) = 0$
(d) $(3t - 2)(t + 3) = 0$
(e) $(2x - 1)(3x - 2) = 0$
(f) $(r - 3)(r + 2) = 0$
(g) $a(a - 5) = 0$
(h) $(4x + 3)(5x - 2) = 0$
(i) $(3 - 4p)(2 - 7p) = 0$

2. Factor each expression. Then find the roots of each equation.

(a) $n^2 + 7n - 30 = 0$
(b) $2y^2 + 9y + 4 = 0$
(c) $m^2 + 8m + 15 = 0$
(d) $y^2 - y - 6 = 0$
(e) $x^2 - 2x - 15 = 0$
(f) $m^2 - 1 = 0$
(g) $4n^2 - 1 = 0$
(h) $16x^2 - 25 = 0$
(i) $9n^2 - 6n + 1 = 0$
(j) $3x^2 + 9x - 30 = 0$
(k) $5x^2 + 25x + 30 = 0$
(l) $4x^2 - 14x - 8 = 0$

3. Solve each equation.

(a) $42 = x^2 - x$
(b) $x^2 - 4x = 21$
(c) $a^2 = 2a + 48$
(d) $m^2 = 30 - 7m$
(e) $3 = 6x^2 - 7x$
(f) $15 + x = 2x^2$
(g) $2y^2 + 4 = -9y$
(h) $17x + 5x^2 = -6$
(i) $2m^2 = 3 - 5m$

4. Wanda and Louise determine that the expression $A = -2w^2 + 36w$ models the area of a rectangular puppy run, where w is the width in metres and A is the area in square metres. What dimensions produce an area of 112 m^2?

5. Mirna's Fashion store determined that each relation below modelled expected revenue for an article of clothing. R is the revenue in dollars and x is the amount of the change in price. Solve for x. Interpret each answer in the context of the question.

(a) Sweatshirts $R = (20 - 2x)(30 + 3x)$ when $R = \$594$

(b) Suits $R = (200 - 25x)(9 + 3x)$ when $R = \$2250$

(c) Pants $R = (30 - 4x)(16 + 2x)$ when $R = \$396$

B

6. Determine all values of x that satisfy the equation.

(a) $y = x^2 - x - 30$, when $y = -24$

(b) $y = x^2 - 3x - 28$, when $y = -10$

(c) $y = x^2 - 7x + 12$, when $y = 2$

(d) $y = 2x^2 - 9x + 10$, when $y = -6$

(e) $y = 6x^2 - x - 1$, when $y = 50$

(f) $y = x^2 - 8x + 16$, when $y = 4$

7. Graph each relation in question 6 using a graphing calculator and confirm the values of x that you found.

8. A pair of skydivers jump out of an airplane 5.5 km above the ground. The equation $H = 5500 - 5t^2$ is an approximate model for the divers' altitude in metres at t seconds after jumping out of the plane.

(a) After 10 s how far have the divers fallen?

(b) They open their chutes at an altitude of 1000 m. How long did they free-fall?

(c) If a parachute does not open at 1000 m, how much time is left to use the emergency chute?

9. A professional stunt performer at a theme park dives off a tower 21 m high into the water. His height above the ground at time t seconds is given by the equation $h = -4.9t^2 + 21$.

(a) How long does it take to reach the halfway mark?

(b) How long does it take to reach the water?

(c) Compare the times in (a) and (b). Explain why the time at the bottom is not twice the time at the halfway point.

10. Sasha works at the Botanical Gardens and is planning a new rectangular rose garden 20 m by 30 m. She plans to build a walkway, with a uniform width, all around the garden. Her budget is \$6000 and she knows it will cost \$10/m^2 to construct the walkway. How wide can the walkway be?

11. Sasha's budget in question 10 is reduced by \$1000. How will this affect the width of the walkway?

12. **Knowledge and Understanding:** Suppose that the population of a town is described by $P = 0.16t^2 + 7.2t + 100$, where P is the population in thousands and t is the time in years, with $t = 0$ representing the year 2000.
   (a) What will the population be in 2010?
   (b) What was the population in 1995?
   (c) When will the population reach 52 000?
   (d) Will the population ever reach zero under this model? Explain.

13. A model rocket is shot straight up from the roof of a school. The height at any time t is approximated by the model $H = 15 + 23t - 5t^2$, where H is the height in metres and t is the time in seconds.
   (a) What is the height of the school?
   (b) How long does it take for the rocket to pass a window 10 m above the ground?
   (c) When does the rocket hit the ground?
   (d) What is the maximum height the rocket reaches above the roof of the school?

14. **Communication:** Water from a fire hose is sprayed on a fire 15 m up the side of a wall. The equation $H = -0.011x^2 + x + 1.6$ models the height of the jet of water and the horizontal distance from the nozzle in metres. What is the farthest distance back from the building that a firefighter could stand and still reach the fire? Explain. Include a diagram with your explanation.

15. The safe stopping distance, d, in metres, for a boat travelling at v km/h in calm water is determined to be $d = 0.002(2v^2 + 10v + 3000)$.
   (a) What is the safe stopping distance if the speed is 12 km/h?
   (b) What is the initial speed of the boat if it takes 15 m to stop?

16. **Application:** A Zamboni is resurfacing the ice on a rectangular ice rink 25 m wide by 40 m long. The operator starts at the centre of the rink and makes uniform rectangular passes over the surface, gradually working outward in a concentric way. After only ten percent of the surface is complete, the machine breaks down. Find the width of the remaining strip of unfinished ice surface.

17. Nancy walks 15 m diagonally across a rectangular field. She then returns to her starting position along the outside of the field. The total distance she walks is 36 m. What are the dimensions of the field?

18. Raj and his sister cut the lawn. The lawn is a square and Raj says he will cut a path around the outside 3 m wide. His sister will cut the remaining lawn. If the part he cuts is the same area as the part she cuts, what are the dimensions of the lawn?

19. **Check Your Understanding**

(a) Explain in detail how to determine the value of the independent variable in a quadratic relation if the value of the dependent variable is known.

(b) What is the greatest number of solutions a quadratic equation can have? Explain, with an example, why all of the solutions to the equation may not be reasonable answers to the original problem.

C

20. **Thinking, Inquiry, Problem Solving**

Nicole is doing a project on the planets in the solar system. She wants to show the different effect of gravity on each planet and decides to examine the flight of an imaginary arrow that is shot straight up from the surface of the planet.

The table shows how the pull of gravity on each planet compares with gravity on Earth.

Planet	Relative Gravity (% of Earth's Gravity, g)
Earth ($g = -9.8$ m/s^2)	100%
Mercury	38%
Venus	81%
Mars	40%
Jupiter	254%
Saturn	108%
Uranus	91%
Neptune	190%
Pluto	8%

The equation $H = 2.3 + 50t + \frac{1}{2}gt^2$ models the height of the arrow in metres at time t seconds. The pull of gravity is g and is different for each planet. On Earth, $g = -9.8$ m/s^2.

(a) Determine the equation that models the height of the arrow at time t on each planet.

(b) What is the maximum height the arrow would reach on each planet?

(c) How long would the arrow be in flight on each planet?

(d) At what time would the arrow reach a distance halfway between the ground and its maximum height?

(e) Account for the different forces of gravity on each planet.

21. A new children's play area will be a square 40 m by 40 m. Inside the playground will be four square sand boxes each with a side length of 4 m. How must they be placed so that the distance between the sand boxes and between the outside boundary and the sand boxes is the same?