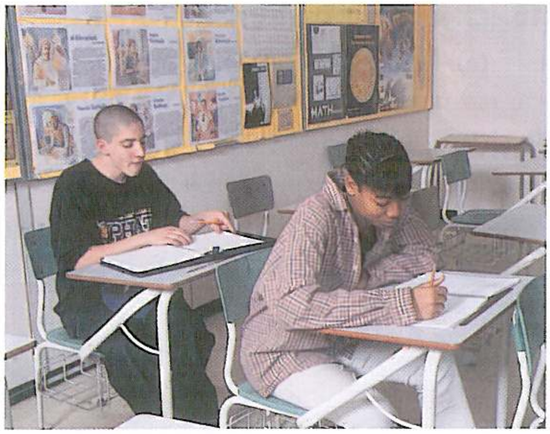
3.8 Extending Algebra Skills: Factoring Quadratic Expressions

Part 1: Factoring a Quadratic Expression in Standard Form



Earlier in this chapter, you learned that having a quadratic relation in factored form permits you to quickly find the zeros and the vertex of the parabola. Factored form is very useful for modelling situations with quadratic expressions and for solving maximum or minimum problems.

You saw that a factored expression like $(2x - 5)(3x + 2)$ can be expanded using the distributive property. The resulting quadratic expression is said to be in standard or expanded form.

You also used the distributive property to find missing values in expressions like

$(x + 5)(x - \blacksquare) = x^2 + \blacksquare x - 15$

What if you start with a quadratic expression in standard form? Can you use the distributive property to convert it to factored form?

Think, Do, Discuss

1. Consider the quadratic expression $x^2 + 8x + 15$.

- **(a)** If the expression can be factored, will the form be $(x - s)(x - t)$ or $a(x - s)(x - t)$?
- **(b)** Apply the distributive property. What does the result tell you about the value of st? of $s + t$?
- **(c)** Explain how you can use the results from (b) to find the values of s and t. What are the values?
- **(d)** Verify the values of s and t by graphing the relation $y = x^2 + 8x + 15$.
- **(e)** Explain how the graph verifies the factored form.

2. **(a)** Explain why the expression $x^2 + 8x + 20$ cannot be factored.

- **(b)** Does this mean that the expression has no zeros? How do you know?

3. Think about the expression $6x^2 + 24x - 72$.

(a) If this expression can be factored, the factored form must be either $a(x - s)(x - t)$ or $(ax - s)(bx - t)$. How can you tell that the form $a(x - s)(x - t)$ is most likely to work?

(b) If the form is $a(x - s)(x - t)$, what must the value of a be?

(c) What must the value of st be? Why?

(d) What information does the coefficient of x tell you about s and t?

(e) Use the results from (b), (c), and (d) to find the factors of the expression.

(f) How is factoring this expression similar to factoring the expression in question 1? How is it different?

4. Consider the quadratic expression $6x^2 + 25x - 9$.

(a) If this expression can be factored, the factored form must be either $a(x - s)(x - t)$ or $(ax - s)(bx - t)$. How can you determine that the form $(ax - s)(bx - t)$ is most likely to work?

(b) If the form is $(ax - s)(bx - t)$, what must ab and st be equal to?

(c) What information does the coefficient of x tell you about a, b, s, and t?

(d) Use all the information from (b) and (c) to find the factors using a guess-and-check strategy.

Part 2: Some Special Factoring Patterns

Some types of products reveal special patterns when they are expanded using the distributive property. If you can recognize these special patterns, it will be much easier to factor many polynomial expressions.

Think, Do, Discuss

1. Use the following expressions for this question.

i. $(a + 1)(a - 1)$	**ii.** $(b + 2)(b - 2)$	**iii.** $(c - 10)(c + 10)$
iv. $(2d + 1)(2d - 1)$	**v.** $(3e + 5)(3e - 5)$	**vi.** $(6 - 5f)(6 + 5f)$

(a) Apply the distributive property to expand each expression. Describe the pattern that all of the expanded products have in common.

(b) Explain why the phrase "difference of squares" is a good way to describe this pattern.

(c) If the trinomial $x^2 + ax + b$ can be factored using the same pattern, what must be true about the values of a and b?

(d) If the trinomial $ax^2 + bx + c$ can be factored using the same pattern, what must be true about the values of a, b, and c?

2. Examine how the following products can be expanded using the distributive property.

   i. $(a + 1)(a + 1)$ ii. $(b - 2)(b - 2)$ iii. $(c - 10)^2$

   iv. $(2d + 1)^2$ v. $(3e - 5)^2$ vi. $(6 - 5f)^2$

   (a) Describe the pattern that all of the expanded products have in common.

   (b) Explain why the phrase "perfect square" is a good way to describe this product pattern.

   (c) If the trinomial $x^2 + ax + b$ can be factored using the same pattern, what must be true about the values of a and b?

   (d) If the trinomial $ax^2 + bx + c$ can be factored using the same pattern, what must be true about the values of a, b, and c?

Focus 3.8

Key Ideas

- Many quadratic relations in standard form $y = ax^2 + bx + c$ can be expressed as the product of two binomial factors. Finding this product is called **factoring.**
- Factoring is the opposite operation of expanding.
- If the quadratic expression $x^2 + bx + c$ can be factored, the factors are of the form $(x - s)(x - t)$, where $b = -(s + t)$ and $c = st$.
- If the quadratic expression $ax^2 + bx + c$ (where $a \neq 1$) can be factored, the factors can be found using a guess-and-check strategy.
  - ♦ Choose factors that produce the correct first and last term of the quadratic (the x^2-term and the constant term) when multiplied.
  - ♦ The middle (x) term of the quadratic comes from multiplying the outside terms of the binomial factors together, then the inside terms, then adding the results. For example, $6x^2 - 23x + 20 = (3x - 4)(2x - 5)$, since $(3x)(2x) = 6x^2$, $(-4)(-5) = 20$, and $(3x)(-5) + (-4)(2x) = (-15x) + (-8x)$ or $-23x$. Check that the factors you choose give the correct x-term in the standard form.
  - ♦ Sometimes quadratics in the form $ax^2 + bx + c$ have a common factor. If the trinomial remaining after a common factor is removed is in the form $x^2 + bx + c$, you may be able to factor it as above. For example, $5x^2 - 5x - 150 = 5(x - 6)(x + 5)$, since $5x^2 - 5x - 150 = 5(x^2 - x - 30)$ and $(x^2 - x - 30) = (x - 6)(x + 5)$.
- If a trinomial is in the form $a^2x^2 - b^2$, then it is a **difference of squares** and can be factored as $(ax + b)(ax - b)$.
- If a trinomial is in the form $a^2x^2 + 2abx + b^2$, then it is a **perfect square** and can be factored as $(ax + b)(ax + b) = (ax + b)^2$.

Example 1

Factor each expression.

(a) $x^2 - 14x + 45$

(b) $4x^2 + 12x - 40$

Solution

(a) The trinomial $x^2 - 14x + 45$ can be factored if there are two numbers which multiply to give 45 and add to give -14.

Using guess-and-check, the numbers -5 and -9 are found to satisfy these conditions, since $(-5)(-9) = 45$ and $(-5) + (-9) = -14$.

Therefore, $x^2 - 14x + 45 = (x - 5)(x - 9)$.

(b) The trinomial $4x^2 + 12x - 40$ has a common factor, 4, in each term.

$4x^2 + 12x - 40$	Remove the common factor.
$= 4(x^2 + 3x - 10)$	Find two numbers that multiply to give -10 and
$= 4(x + 5)(x - 2)$	add to give 3. The numbers 5 and -2 work, since $(5)(-2) = -10$ and $(5) + (-2) = 3$.

Example 2

Factor each expression.

(a) $x^2 - 9$ **(b)** $4x^2 - 12x + 9$ **(c)** $3x^2 + 8x + 4$

Solution

(a) $x^2 - 9 = (x + 3)(x - 3)$

You can recognize this expression as a difference of squares and factor it immediately using the pattern discussed above.

$$x^2 - 9 = x^2 + 0x - 9$$
$$= (x + 3)(x - 3)$$

Or you can use the general approach and look for two numbers that multiply to give -9 and add to give 0, since the coefficient of the x-term is 0. The required numbers are 3 and -3.

(b) $4x^2 - 12x + 9$

The coefficient of the x^2-term is not 1 and there is no common factor. The factors of the trinomial will be two binomials. Try visualizing the distributive property of two binomials to determine the factors.

The first term and last term are perfect squares:

$$4x^2 = (2x)(2x) \text{ and } 9 = (3)(3) \text{ or } (-3)(-3)$$

The middle term is composed of the factors of the first and last terms:

$-12x = (2x)(-3) + (2x)(-3)$ Select the -3, so the sum of products is negative.

Following the pattern discussed above, this expression is a perfect square.

$$\begin{aligned} 4x^2 - 12x + 9 &= (2x - 3)(2x - 3) \\ &= (2x - 3)^2 \end{aligned}$$

Check: $$\begin{aligned} (2x - 3)(2x - 3) &= 4x^2 - 6x - 6x + 9 \\ &= 4x^2 - 12x + 9 \end{aligned}$$

(c) $3x^2 + 8x + 4$

The coefficient of the x^2-term is not 1 and there is no common factor. The factors of the trinomial will be two binomials. Try visualizing the distributive property of two binomials to determine the factors.

First term: $3x^2 = (3x)(x)$

Last term: $4 = (2)(2)$ or $(-2)(-2)$

The middle term is composed of the factors of the first and last terms:

$$\begin{aligned} 8x &= (3x)(2) + (2)(x) \\ 3x^2 + 8x + 4 &= (3x + 2)(x + 2) \end{aligned}$$

Select 2 so the sum of the products is positive.

Check: $$\begin{aligned} (3x + 2)(x + 2) &= 3x^2 + 6x + 2x + 4 \\ &= 3x^2 + 8x + 4 \end{aligned}$$

Example 3

For a set of experimental data, the QuadReg function of a graphing calculator has given the curve of best fit as $y = -2x^2 + 8x + 42$. Find the vertex of the parabola.

Solution

The expression has a common factor, -2. Remove it to give $y = -2(x^2 - 4x - 21)$. Now try to factor the trinomial $x^2 - 4x - 21$.

To factor as $(x - s)(x - t) = x^2 - 4x - 21$, you must find s and t such that $st = -21$ and $s + t = 4$. Two numbers that multiply to -21 and add to 4 are 7 and -3.

Therefore, $x^2 - 4x - 21 = (x - 7)(x + 3)$.

The factored form of the relation is $y = -2(x - 7)(x + 3)$.

The vertex is on the perpendicular bisector of the zeros. Since the zeros are at $x = 7$ and $x = -3$, the vertex occurs on the line $x = \frac{7 - 3}{2}$, or $x = 2$.

Substitute $x = 2$ in the equation to find the y-coordinate of the vertex.

$$\begin{aligned} y &= -2(x - 7)(x + 3) \\ y &= -2(2 - 7)(2 + 3) \\ y &= -2(-5)(5) \\ y &= 50 \end{aligned}$$

The vertex of the parabola is at (2, 50).

Practise, Apply, Solve 3.8

A

1. Find two numbers with these properties.

(a) product is 56 and sum is 15

(b) product is -16 and sum is -6

(c) product is -12 and sum is -1

(d) product is -35 and the sum is 2

2. Factor each expression.

(a) $x^2 + 3x + 2$
(b) $x^2 + 5x + 4$
(c) $f^2 - 6f + 9$
(d) $c^2 + 2c - 15$
(e) $g^2 + 3g - 18$
(f) $r^2 - 2r - 8$
(g) $m^2 - 5m - 14$
(h) $n^2 - 9n + 20$
(i) $x^2 - 10x + 16$
(j) $a^2 + 6a + 9$
(k) $x^2 - 8x + 15$
(l) $y^2 + 8y + 16$
(m) $x^2 + 5x - 36$
(n) $b^2 - 4b - 32$
(o) $x^2 - 15x + 56$
(p) $v^2 + 6v - 27$
(q) $t^2 + 2t - 48$
(r) $p^2 - 17p + 72$

B

3. Factor each expression. Remember to look for common factors first.

(a) $3x^2 + 24x + 45$
(b) $2y^2 - 2y - 60$
(c) $3a^2 + 9a + 6$
(d) $5x^2 - 10x + 5$
(e) $6x^2 + 24x - 30$
(f) $x^3 + 5x^2 + 4x$
(g) $8m^2 - 104m + 336$
(h) $21x^2 + 21x - 42$
(i) $7x^2 + 28x - 147$

4. Factor each expression.

(a) $x^2 - 25$
(b) $c^2 - 49$
(c) $a^2 - 36$
(d) $x^2 - 81$
(e) $d^2 - 121$
(f) $b^2 - 64$
(g) $9x^2 - 4$
(h) $64a^2 - 1$
(i) $25p^2 - 49$
(j) $16c^2 - 81$
(k) $50r^2 - 72$
(l) $7y^2 - 28$

5. Factor each expression.

(a) $9x^2 - 6x + 1$
(b) $25x^2 + 20x + 4$
(c) $4a^2 - 20a + 25$
(d) $49c^2 + 42c + 9$
(e) $100x^2 - 180x + 81$
(f) $36g^2 + 60g + 25$
(g) $9v^2 - 12v + 4$
(h) $64c^2 + 16c + 1$
(i) $16d^2 - 24d + 9$

6. Factor each expression.

(a) $2t^2 + t - 6$
(b) $3m^2 - 11m - 4$
(c) $10x^2 + 3x - 1$
(d) $9x^2 + 12x + 4$
(e) $9x^2 - 12x + 4$
(f) $4x^2 - 16x + 15$
(g) $2y^2 + 3y + 1$
(h) $3b^2 - 5b - 2$
(i) $2c^2 + 5c - 12$
(j) $6x^2 + 5x + 1$
(k) $5a^2 - 11a + 2$
(l) $6m^2 - 11m - 10$
(m) $2d^2 + 5d + 2$
(n) $6w^2 - 13w + 6$
(o) $10b^2 + b - 3$

7. Factor each expression.

(a) $3a^2 + 6a$	(b) $2x - 8xy$	(c) $25a^2 - 9$
(d) $x^2 + 7x + 12$	(e) $y^2 - 11y + 28$	(f) $16a^2 - 8a + 1$
(g) $8 + 6x + x^2$	(h) $5b^2 - 14b + 8$	(i) $10x^2 - 28x + 16$
(j) $3d^2 - 432$	(k) $6d^2 + 5d + 1$	(l) $56c^2 + 9c - 2$
(m) $2g^2 - 2g - 24$	(n) $-16 + 9x^2$	(o) $x^2y^3z - 2xy^2$

8. **Knowledge and Understanding:** For each relation

i. express it in factored form

ii. determine its zeros

iii. determine the coordinates of its vertex

iv. graph the relation

(a) $y = x^2 - 4$	(b) $y = x^2 + 6x + 8$	(c) $y = x^2 - 6x + 5$
(d) $y = -x^2 + 2x + 24$	(e) $y = x^2 + 2x + 1$	(f) $y = -x^2 + 3x + 18$

9. A rectangular enclosure has an area in square metres given by $A = -2w^2 + 36w$, where w is the width of the rectangle in metres. What is the maximum area of the enclosure?

10. **Communication:** Can all equations of parabolas be expressed in factored form? Explain.

11. A model rocket is shot into the air and its path is approximated by $h = -5t^2 + 30t$, where h is the height of the rocket above the ground in metres and t is the elapsed time in seconds.

(a) When will the rocket hit the ground?

(b) What is the maximum height of the rocket?

12. A baseball is thrown from the top of a building and falls to the ground below. Its path is approximated by the relation $h = -5t^2 + 5t + 30$, where h is the height above ground in metres and t is the elapsed time in seconds.

(a) How tall is the building?

(b) When will the ball hit the ground?

(c) When does the ball reach its maximum height?

(d) How high above the building is the ball at its maximum height?

13. **Application:** A small company that manufactures snowboards uses the relation $P = 162x - 81x^2$ to model its profit. In the model, x represents the number of snowboards in thousands, and P represents the profit in thousands of dollars.

(a) What is the maximum profit the company can earn?

(b) How many snowboards must it produce to earn this profit?

(c) The company breaks even when there is neither a profit nor a loss. What are the break-even points for the company?

14. A computer software company models the profit on its latest game using the relation $P = -2x^2 + 28x - 90$, where x is the number of games it produces in hundred thousands and P is the profit in millions of dollars.

(a) What is the maximum profit the company can earn?

(b) How many games must it produce to earn this profit?

(c) What are the break-even points for the company?

15. **Check Your Understanding**

(a) Explain how to change a quadratic relation from standard form into factored form.

(b) Describe the advantages of working with a quadratic relation in factored form compared with standard form.

Ⓒ

16. The path of a shot put is given by $h = -0.0502(d^2 - 20.7d - 26.28)$ where h is the height and d is the horizontal distance in metres.

(a) Rewrite the relation in the form $h = a(d - s)(d - t)$ where s and t are the zeros of the relation.

(b) What is the significance of s and t in this question?

17. Factor completely. Identify the expressions that cannot be factored.

(a) $4x^4 - 36x^2 + 9$ (b) $4x^4 + 24x^2 + 9$

(c) $8x^2 - 50$ (d) $\frac{a^2}{64} - \frac{b^2}{49}$

(e) $\frac{c^4}{16} - \frac{d^4}{81}$ (f) $625m^8n^4 - 16p^8$

(g) $100 - (w - 4)^2$ (h) $4x^2 + y^2$ (i) $x^2 - 2xy + y^2 - 9z^2$

(j) $a^2 + 6a + 9 - b^2$ (k) $4a^2b^2 + 12abc + 9c^2$ (l) $1 - 6x + 9x^2 - 4y^2$

18. **Thinking, Inquiry, Problem Solving:** Soundz Inc. makes CD players. Last year, accountants modelled the company's profit by $P = -5x^2 + 60x - 135$. Over the course of the year, in an effort to become more efficient, Soundz Inc. restructured its operation, eliminating some employees and reducing costs. This year, accountants are using $P = -7x^2 + 70x - 63$ to project the company's profit. In both models, P is the profit in hundreds of thousands of dollars and x is the number of CD players made, in hundreds of thousands. Was Soundz Inc.'s restructuring effective? Justify your answer.

The Chapter Problem—Setting the Best Ticket Price

In this section you changed quadratic relations from standard form into factored form. Apply what you learned to answer these questions about the chapter problem on page 242.

1. The relation $R = 4000 + 20x - 2x^2$, where x is the amount of the price change and R is the revenue in dollars, is a model that predicts revenue from ticket sales. What is the maximum revenue predicted by this model?

2. Revisit section 3.1 and use the table of values and the quadratic regression feature of a graphing calculator to draw the scatter plots and the curves of best fit. Compare these graphs to the hand-drawn graphs. How can they be used to solve the original problem?