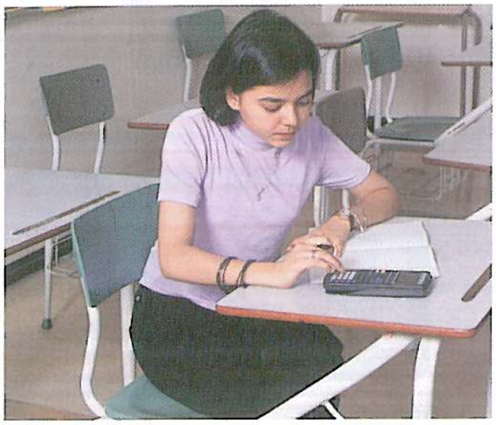
3.7 Standard Form of a Quadratic Relation

Part 1: More About the Balloon Catapult

Recall the projectile motion problem from section 3.4 on page 275. The projectile was a water balloon launched from a catapult.

The graph was drawn with a graphing program, using data from a stop-motion photograph of the balloon's flight. The table of values is reproduced below.

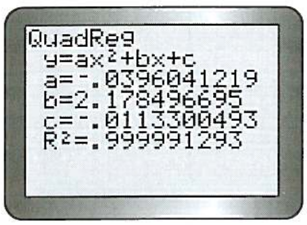


Horizontal Distance (m)	Height (m)	Horizontal Distance (m)	Height (m)
0	0.0	28	29.9
2	4.2	30	29.7
4	8.1	32	29.1
6	11.6	34	28.3
8	14.9	36	27.1
10	17.8	38	25.6
12	20.4	40	23.8
14	22.7	42	21.6
16	24.7	44	19.2
18	26.4	46	16.4
20	27.7	48	13.3
22	28.7	50	9.9
24	29.5	52	6.2
26	29.9	54	2.1

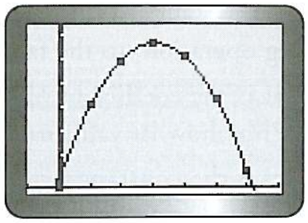
Think, Do, Discuss

1. **(a)** Enter the data into the lists of a graphing calculator.

(b) Use the quadratic regression feature of the calculator to determine the algebraic expression for the curve of best fit for this data. Remember to activate DiagnosticOn if need be. If necessary, refer to page 106 of section 1.10. Your display should look similar to the sample shown. This form of a quadratic relation, $y = ax^2 + bx + c$, is called **standard form.**



(c) Plot this quadratic relation and comment on how closely the curve fits the data.



(d) Examine the value of R^2. Give some reasons why the fit might not be exact.

2. Working from the graph in section 3.4, one group of students came up with the equation $y = -0.04x(x - 55)$ for the same situation.

   (a) Demonstrate that this equation and the equation found in step 1 are both quadratic models for the same situation.

   (b) Use both algebraic models to test the value for the maximum height of the balloon.

   (c) Which produces a value closer to the value in the original graph in section 3.4?

   (d) Check how well each model predicts values from the original graph or from the table of values above. Use several different data points.

   (e) Which algebraic expression is a more accurate model of the flight trajectory of the balloon?

   (f) Is one model better than the other? Explain.

Part 2: Comparing Standard Form and Factored Form

Can an equation in factored form, $y = a(x - s)(x - t)$, be equivalent to a quadratic equation in standard form, $y = ax^2 + bx + c$?

Think, Do, Discuss

1. In section 3.4, Mirna used the quadratic relation $R = (10 - 0.1x)(30 + x)$ as a model for her revenue from T-shirt sales.

   (a) Graph the factored form of the relation.

   (b) Use this equation to create a table of values for the relation with at least 10 ordered pairs.

   (c) Enter the ordered pairs into the list editor of a graphing calculator, perform a quadratic regression, and graph the result.

   (d) Compare the graph in (a) with the graph of the quadratic regression relation in (c). How closely do they match?

2. Compare the standard form of the equation, produced by the calculator's QuadReg operation, to the factored form.

   (a) What is the coefficient of the x^2-term in the standard form of the equation? Describe how its value can be computed from the terms in the factored form.

   (b) What is the constant term in the standard form of the equation? Describe how its value can be computed from the terms in the factored form.

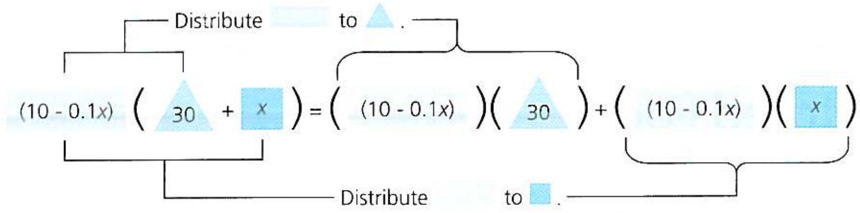
   (c) What is the coefficient of the x-term in the standard form of the equation? Describe how it can be calculated from the factored form.

Part 3: Expanding and Simplifying Using the Distributive Property

In your previous mathematics studies, you learned about the distributive property for multiplying a polynomial and a monomial. This property can be used to convert the factored form of a quadratic relation to standard form.

Think, Do, Discuss

1. Recall the factored form of Mirna's T-shirt relation, $R = (10 - 0.1x)(30 + x)$. Rewrite the factors as shown.



   (a) Describe how the distributive property lets you multiply the first binomial, $(10 - 0.1x)$, by each term in the second binomial.

   (b) Examine the two products on the right side of the equation in the diagram. Explain how applying the distributive property again allows you to expand each of those products.

   (c) Simplify the expression by collecting like terms.

   (d) Write the final expanded and simplified form of $R = (10 - 0.1x)(30 + x)$.

   (e) Explain how you could use the steps above to express the relation $R = 2.5(10 - 0.1x)(30 + x)$ in standard form.

2. Recall Max's CD store from section 3.4. The quadratic relation Max used as a model for revenue from CD sales was $R = (20 + 0.5x)(280 - 5x)$. Follow the process in step 1 to rewrite this expression in standard form.

Focus 3.7

Key Ideas

- The **factored form** of a quadratic relation, $y = (x - s)(x - t)$, $y = a(x - s)(x - t)$, or $y = (ax - s)(bx - t)$, can be expanded using the distributive property and simplified to give a **standard form** trinomial, $y = ax^2 + bx + c$. Standard form is often called **expanded form**.
- Equivalent factored and standard forms

Factored Form	Standard (or Expanded) Form
$y = (x - s)(x - t)$	$y = x^2 - (s + t)x + st$
$y = a(x - s)(x - t)$	$y = ax^2 - a(s + t)x + ast$
$y = (ax - s)(bx - t)$	$y = abx^2 - (at + bs)x + st$

- The **quadratic regression** feature of a graphing calculator provides the algebraic expression for a curve of best fit in **standard** (or **expanded**) **form**, $y = ax^2 + bx + c$.

Example 1

Expand these expressions.

(a) $(x + 3)(x - 4)$

(b) $(x - 4)^2$

(c) $(x - 5)(x + 5)$

(d) $(2x + 3)(3x - 7)$

Solution

(a) $(x + 3)(x - 4)$ — Apply the distributive property.

$= (x + 3)(x) + (x + 3)(-4)$ — Multiply $(x + 3)$ by x and $(x + 3)$ by -4.

$= x^2 + 3x + x(-4) + 3(-4)$ — Multiply.

$= x^2 + 3x - 4x - 12$ — Collect like terms.

$= x^2 - x - 12$ — Simplify.

(b) $(x - 4)^2$

$= (x - 4)(x - 4)$ — Apply the distributive property.

$= (x - 4)(x) + (x - 4)(-4)$ — Multiply $(x - 4)$ by x and $(x - 4)$ by -4.

$= x^2 - 4x + x(-4) - 4(-4)$ — Multiply.

$= x^2 - 4x - 4x + 16$ — Collect like terms.

$= x^2 - 8x + 16$ — Simplify.

(c) $(x - 5)(x + 5)$	Apply the distributive property.
$= (x - 5)(x) + (x - 5)(5)$	Multiply $(x - 5)$ by x and by 5.
$= x^2 - 5x + 5x - 25$	Multiply.
$= x^2 + 0x - 25$	Collect like terms.
$= x^2 - 25$	Simplify.
(d) $(2x + 3)(3x - 7)$	Apply the distributive property.
$= (2x + 3)(3x) + (2x + 3)(-7)$	Multiply $(2x + 3)$ by $3x$ and by -7.
$= 2x(3x) + 3(3x) - 2x(7) + 3(-7)$	Multiply.
$= 6x^2 + 9x - 14x - 21$	Collect like terms.
$= 6x^2 - 5x - 21$	Simplify.

Example 2

Find the values of a and b.

$(x + a)(x + 3) = x^2 + 5x + b$

Solution

Expand the left side using the distributive property.

$$x^2 + ax + 3x + 3a = x^2 + 5x + b$$

$$x^2 + (3 + a)x + 3a = x^2 + 5x + b \qquad \text{Collect like terms on the left side.}$$

The coefficients of corresponding like terms must be equal, so

$$3 + a = 5 \qquad \text{(from the } x\text{-terms)}$$

which gives $a = 2$

and $3a = b$ (from the constant terms)

Substituting $a = 2$ gives $3(2) = b$ or $b = 6$.

Therefore, $a = 2$ and $b = 6$.

Example 3

Find the standard form algebraic model that represents a parabola with zeros at 2 and -4 and a y-intercept of 16.

Solution

In factored form, the quadratic relation describing the parabola must be

$$y = a(x - 2)(x + 4)$$

Since the y-intercept is 16, you can substitute (0, 16) into this equation and solve for a.

$$y = a(x - 2)(x + 4)$$
$$16 = a(0 - 2)(0 + 4)$$
$$16 = -8a$$
$$\frac{16}{-8} = \frac{-8a}{-8}$$
$$-2 = a$$

The factored form of the relation is $y = -2(x - 2)(x + 4)$.

Apply the distributive property to convert the equation to standard form.

$y = -2(x - 2)(x + 4)$

$y = -2[(x - 2)(x + 4)]$ Multiply the binomials together.

$y = -2[(x - 2)x + (x - 2)4]$ Expand.

$y = -2(x^2 - 2x + 4x - 8)$ Simplify.

$y = -2(x^2 + 2x - 8)$ Multiply by -2.

$y = -2x^2 - 4x + 16$

Practise, Apply, Solve 3.7

A

1. Fill in the missing terms.

(a) $(m + 3)(m + 2) = ■ + 3m + 2m + ■$

(b) $(k - 2)(k + 1) = ■ - 2k + ■ - 2$

(c) $(r + 4)(r - 3) = r^2 + ■ - 3r - ■$

(d) $(x - 5)(x - 2) = x^2 - ■ - ■ + 10$

(e) $(2n + 1)(3n - 2) = ■ + 3n - ■ - 2$

(f) $(5m - 2)(m - 3) = 5m^2 - ■ - 2m + ■$

B

2. Expand and simplify each expression.

(a) $(d + 2)(d + 1)$ **(b)** $(h + 3)(h - 2)$ **(c)** $(p - 3)(p + 4)$

(d) $(a - 2)(a - 3)$ **(e)** $(w - 3)(w - 3)$ **(f)** $(t + 4)(t + 4)$

3. Expand and simplify each expression.

(a) $(2n + 1)(n + 3)$ **(b)** $(q - 2)(3q + 1)$ **(c)** $(3x + 1)(2x - 1)$

(d) $(5m - 1)(2m + 3)$ **(e)** $(2r + 3)(2r + 3)$ **(f)** $(4m + 2)(4m - 2)$

(g) $2(a + 5)(a - 3)$ **(h)** $-2(m - 4)(m - 3)$ **(i)** $5(7h + 6)(4h - 3)$

(j) $-2(3 - 8f)(7 - 6f)$ **(k)** $3(3h - k)(5h + 2k)$ **(l)** $6(4x + 3y)(5x - 2y)$

4. Expand and simplify each expression.

(a) $(x + 3)^2$ **(b)** $(x - 7)^2$ **(c)** $(x + 8)^2$

(d) $(2x + 1)^2$ **(e)** $(4x - 3)^2$ **(f)** $(5x + 7)^2$

5. Find the unknown value or expression.

(a) $6x^2 - 16x - 6 = (x - 3)(6x + \blacksquare)$ **(b)** $x^2 - 3x - 10 = (x - 5)(x + \blacksquare)$

(c) $x^2 - 6x + 8 = (\blacksquare)(x - 2)$ **(d)** $x^2 - 11x + 30 = (x - 5)(\blacksquare)$

(e) $x^2 + 4x - 21 = (x + 7)(\blacksquare)$ **(f)** $5x^2 - 17x + 6 = (\blacksquare - 3)(\blacksquare - 2)$

(g) $6x^2 - 16x + 6 = (\blacksquare - 2)(\blacksquare - 3)$ **(h)** $6x^2 + 29x - 5 = (\blacksquare - 1)(\blacksquare + 5)$

(i) $2y^2 - 7y - 15 = (y - 5)(\blacksquare)$ **(j)** $6y^2 + 19y + 15 = (\blacksquare)(3y + 5)$

(k) $9x^2 - 25 = (3x + 5)(\blacksquare)$ **(l)** $25x^2 + 10x + 1 = (\blacksquare)(5x + 1)$

6. **Communication:** Explain why it is **not** possible to fill in the unknown values in each of the following.

(a) $(a + 5)(a + \blacksquare) = a^2 + 8a + 10$

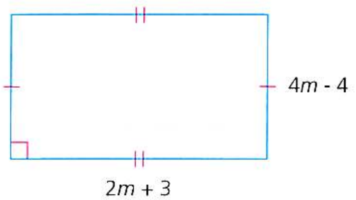
(b) $(b - 2)(b + \blacksquare) = b^2 + 6b - 8$

(c) $(2c + 5)(\blacksquare c + 3) = 6c^2 + 19c + 15$

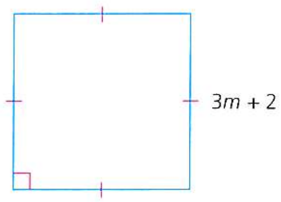
(d) $(2d - 5)(5d + \blacksquare) = 10d^2 - 19d + 15$

7. Write an expression for each area.

(a)

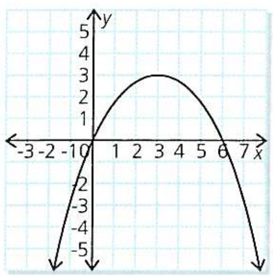


(b)

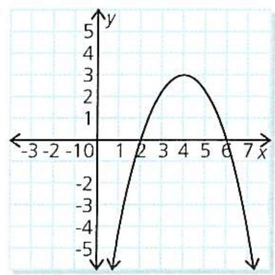


8. Examine each graph. Determine the expanded form of the algebraic relation that defines each parabola.

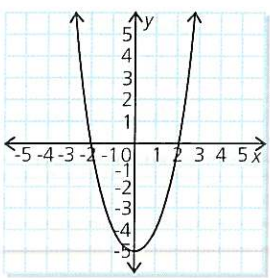
(a)



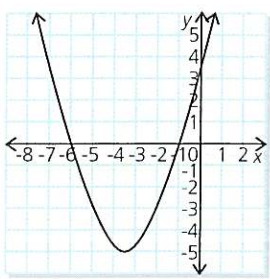
(b)



(c)

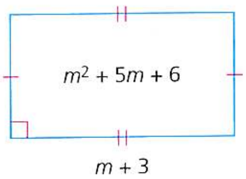


(d)

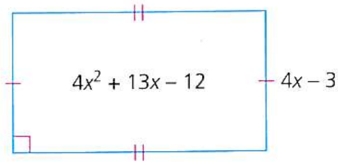


9. The area and one side of a rectangle are given. What is the unknown side?

(a)



(b)

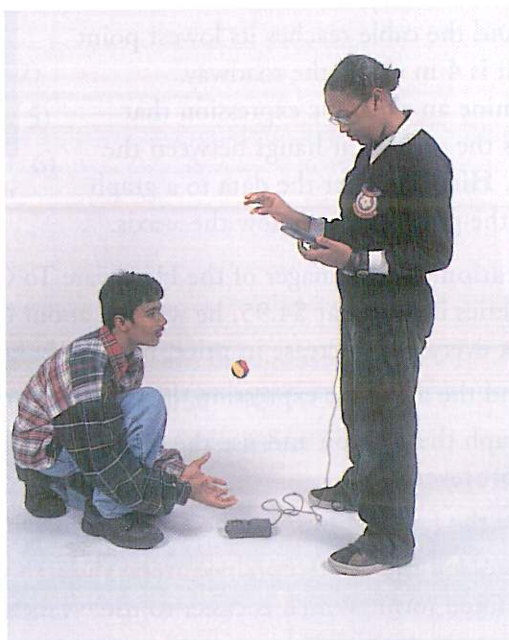


10. For each quadratic relation, write the equation in standard form and tell which way the graph opens.

	Zeros	A Point on the Graph
(a)	–1 and 7	(3, 5)
(b)	–1 and –5	(–3, –4)
(c)	3 and 7	(0, 3)
(d)	–2 and 6	(–1, –1)
(e)	–2 and 8	(3, 7)

11. **Knowledge and Understanding:** Marnie threw a bean bag over a motion detector and it recorded this data.

Time (s)	Height Above Ground (m)
0.00	0.0000
0.25	2.1875
0.50	3.7500
0.75	4.6875
1.00	5.0000
1.25	4.6875
1.50	3.7500
1.75	2.1875
2.00	0.0000



(a) Create a scatter plot and hand draw the curve of best fit.

(b) Determine an algebraic expression that models the data. Express the relation in standard form.

(c) Use a graphing calculator and enter the data into a list. Use quadratic regression to find the equation of the curve of best fit.

(d) Which expression is the better model for the data? Explain.

12. A stone is dropped from a bridge that is 20 m above the river below. This table gives the height of the stone as it falls.

Time (s)	0.00	0.50	1.00	1.50	2.00
Height (m)	20.000	18.775	15.100	8.975	0.400

(a) Create a scatter plot and hand draw the graph of best fit.

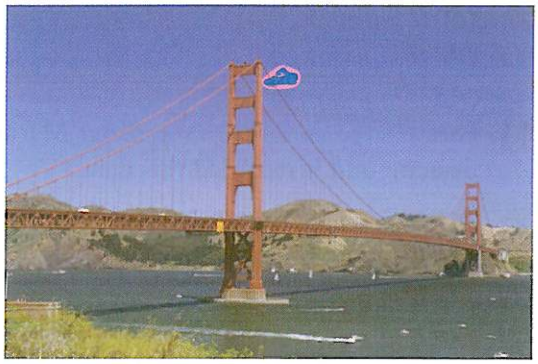
(b) Find the approximate time when the stone hits the water.

(c) Point (0, 20) represents the maximum height of the stone. Use this and the value you found in (b) to approximate the other zero of the relation.

(d) Determine an algebraic expression, in standard form, that models the data.

(e) Use a graphing calculator to determine the quadratic regression equation for the data.

13. The Golden Gate Bridge, in San Francisco, is a suspension bridge. It is supported by a pair of cables that are parabolic in appearance. The cables are attached at either end to a pair of towers at points 152 m above the roadway. The towers are 1280 m apart and the cable reaches its lowest point when it is 4 m above the roadway. Determine an algebraic expression that models the cable as it hangs between the towers. **Hint:** Transfer the data to a graph where the parabola lies below the x-axis.



the Golden Gate Bridge

14. **Application:** The manager of the Hardware To Go store knows that if a package of batteries is priced at \$4.95, he will sell about 600 packages a week. He knows that for every 10¢ increase in price, his weekly sales will decrease by 10 packages.

(a) Find the algebraic expression, in standard form, that models this relation.

(b) Graph the relation and use the graph to find the price that generates the most revenue.

(c) Use the factored form of the expression to find the optimal value.

(d) Compare the values obtained using the standard form plus graph and the factored form. Which is easier to use? Which gives a more accurate answer?

15. Thinking, Inquiry, Problem Solving
The stainless steel Gateway Arch in St. Louis, Missouri, is parabolic in shape. It is 192 m from the base of the north leg to the base of the south leg. The arch is 192 m high. Determine an algebraic expression, in standard form, that models the shape of the arch.


the Gateway Arch

16. Check Your Understanding

(a) Explain how to change a quadratic relation from factored form into standard form.

(b) If you want to sketch the graph of a quadratic relation, which form of the algebraic expression is more helpful, factored form or standard form? Why?

(c) Explain why $y = x^2 + 3x + 4$ cannot be expressed in factored form. What does this tell you about the zeros of this relation?

Ⓒ

17. Expand and simplify each expression.

(a) $(2x + 3)(3x - 1) + x(2x + 4)$

(b) $(2x - 5)(3x + 4) - (4x + 1)(x - 2)$

(c) $(3x - 4)(x + 5) + 2((2x - 3)(x - 5)$

(d) $2(x + 3)(x - 5) - 4(2x + 1)(3x + 6)$

(e) $3(2x + 1)^2 - 5(x - 4)^2$

(f) $-3(2x - 3)^2 + 4(x + 2)^2$

18. Expand and simplify each expression.

(a) $(x + 3)^3$

(b) $(2x - 2)^3$

(c) $(4x + 2y)^3$

(d) $[(x + 2)(x - 2)]^2$

(e) $(x + 6)(x + 3)(x - 6)(x - 3)$

(f) $(3x^2 + 6x - 1)^2$

(g) $(x - 1)^4$

The Chapter Problem—Setting the Best Ticket Price

In this section, you expressed quadratic relations in standard form. Apply what you learned to answer these questions about the chapter problem on page 242.

What is the ticket price that will produce the most revenue for the theatre? In sections 3.2, 3.4, and 3.7, you had enough information to answer this question. Which section provided the most efficient way to find the answer? Explain.