The Role of the Zeros of a Quadratic Relation 3.4

Part 1: A Retailing Problem: Maximizing T-Shirt Revenue

Mirna operates her own store, Mirna's Fashion. A popular style of T-shirt sells for \$10. At that price, Mirna sells about 30 T-shirts a week.

Experience has taught Mirna that changing the price of an article has an effect on sales. For example, she knows that a \$1 increase in the price of the T-shirt means that she will sell about one less T-shirt per week. Similarly, a \$1 decrease in price generally results in one more T-shirt being sold per week. Increasing the price reduces total unit sales, while reducing the price increases unit sales.

Mirna wants to find the price that will maximize her revenue from the sale of T-shirts.

Think, Do, Discuss

1. **(a)** Determine the weekly revenue for several different T-shirt prices.

(b) Set up a table of values that shows how unit price affects the number of T-shirts sold and the total weekly revenue from T-shirt sales.

(c) Use a difference table to determine whether the data would be modelled better by a linear relationship or nonlinear relationship.

(d) List the variables that should be included in an algebraic model of the situation. How will the variables be related?

(e) Write down any relationships that exist between the variables.

(f) Write an algebraic expression to model the total weekly revenue in terms of one of the variables.

2. **(a)** Graph the algebraic relation.

(b) Explain why there are two points on the graph that correspond to a weekly revenue of \$0.

(c) Why does only one of these \$0 points make sense in the context of the problem?

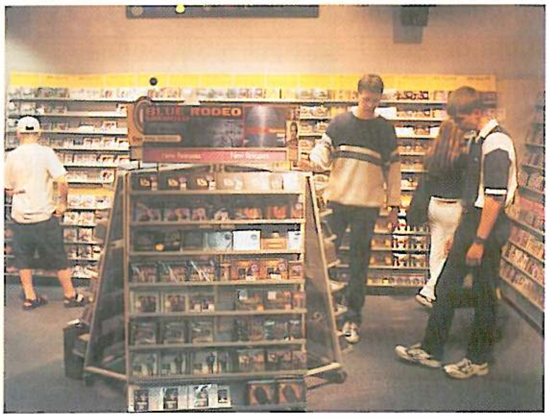
(d) How does knowing both points help you solve the problem?

(e) Use the table of values and the algebraic model to find the unit price that maximizes Mirna's total weekly revenue.

3. Suppose Mirna's original weekly sales were 35 T-shirts instead of 30.
   - (a) Explore how this would affect the price that maximizes her weekly revenue.
   - (b) In this situation, is the graph more or less useful than a table of values? Explain.
4. Suppose the original weekly sales base is still 30 T-shirts, but that each $0.50 change in price (instead of $1) causes a change of one T-shirt in weekly sales.
   - (a) Explore how this would affect the unit price that maximizes weekly revenue.
   - (b) In this situation, is the graph more or less useful than a table of values? Explain.

Part 2: A Retailing Problem: Maximizing CD Revenue

Max operates a store, part of a national chain, that sells CDs in a mall. Max has a one-price-for-all policy: all single CDs sell for $20 each. The national sales manager has given Max permission to change his pricing in an attempt to increase revenue.



Max knows that, over the last six months, he has sold an average of 280 CDs a day at $20 each. The company's market research indicates that for every $0.50 increase in unit price, daily sales will drop by five units.

What unit price will maximize Max's daily revenues?

Think, Do, Discuss

1. (a) Determine the daily revenue for several different CD prices.
   - (b) Set up a table of values that shows how price affects the number of CDs sold and the total daily revenue from CD sales.
   - (c) Use a difference table to determine whether the data would be modelled better by a linear relationship or a nonlinear relationship.
   - (d) List the variables that should be included in an algebraic model of the situation. How will the variables be linked?
   - (e) Write down any relationships that exist between the variables.
   - (f) Write an algebraic expression to model the total weekly revenue in terms of one of the variables.
2. (a) Graph the algebraic relation.
   - (b) Explain why there are two points on the graph that correspond to a daily revenue of $0.

(c) Why does only one of these $0 points make sense in the context of the problem?

(d) How does knowing both points help you solve the problem?

(e) Use the table of values and the algebraic model to find the price that maximizes Max's total daily revenue.

3. Suppose that every $0.25 increase in price causes a corresponding decrease of two CDs in daily sales.

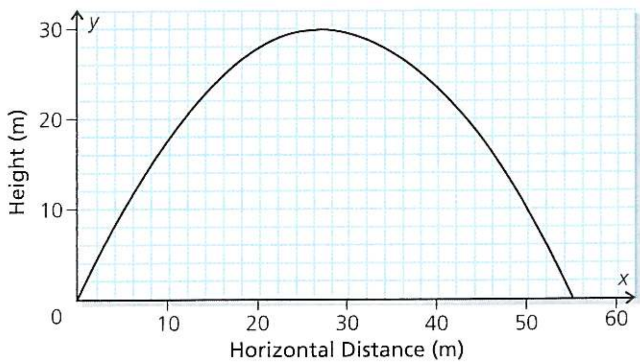
(a) Explore how this would affect the unit price that maximizes daily revenue.

(b) In this situation, is the graph more or less useful than a table of values? Explain.

Part 3: Projectile Motion: Finding the Model —

The graph shows the trajectory of a water balloon launched from a catapult at a recent university Science Day. The objective was to hit a target 55 m from the launch point. The balloon's path was tracked using a stop-motion camera, and the data was then entered into a graphing program that plotted the height of the balloon against its horizontal distance.

Balloon Trajectory—Height vs. Distance



As part of the activity, the engineering students were asked to analyze the graph. They had to find an algebraic model that would predict the height of the balloon at various horizontal distances.

Think, Do, Discuss

1. Use values from the graph to show that the most appropriate model is quadratic.
2. (a) Use the zeros from the graph to write an algebraic expression for the quadratic relation that models the trajectory of the balloon.
   (b) Why is a model that is based only on the zeros not complete?
   (c) What additional information do you need to complete the algebraic expression?
   (d) Modify the algebraic expression of the model by including a variable that represents the missing quantity.
3. Locate a point on the graph whose coordinates can be determined easily and accurately.
   (a) Substitute the coordinates for this point into the partial model in 2(d).
   (b) Solve the resulting equation to find the value of the quantity needed to complete the model.
   (c) Check your model by substituting various values for the independent variable and comparing the predicted results to the graph.
4. (a) Explain why it is possible to find the x-coordinate of the vertex very precisely using your model.
   (b) Why is the y-coordinate of the vertex harder to determine?
   (c) Use the complete model to give a more precise estimate of the maximum height reached by the balloon.

Part 4: Investigating the Graphs of Quadratic Relations

All of the quadratic relations presented so far have been in the form $y = a(x - s)(x - t)$. How do the values of a, s, and t affect the graph?

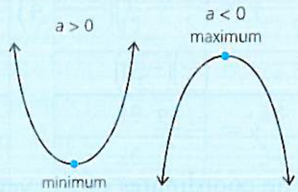
Think, Do, Discuss

1. Use a graphing calculator to graph $y = a(x - 2)(x + 3)$ when $a = 3$. Describe what happens to the graph as you change the value of a to 2, 1, -1, -2, and -3.
2. Graph $y = 2(x - 2)(x - t)$ when $t = -3$. Describe what happens to the graph as you change the value of t to -2, -1, 0, 1, 2, and 3.
3. Which of the quantities a, s, or t affects whether the graph has a maximum or a minimum value? How can you predict if the relation has a maximum or a minimum?
4. What does the equation of a quadratic relation look like if it has two zeros? one zero? no zeros?

Focus 3.4

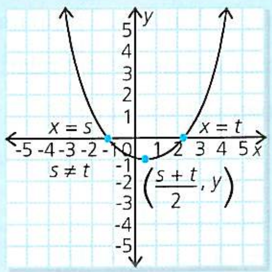
Key Ideas

- Relations in the form $y = a(x - s)(x - t)$ are quadratic, provided that $a \neq 0$.
- A quadratic relation is said to be in **factored form** if its algebraic expression appears in the form $y = a(x - s)(x - t)$.
- If $a > 0$, the parabola opens up and has a minimum.
  If $a < 0$, the parabola opens down and has a maximum.

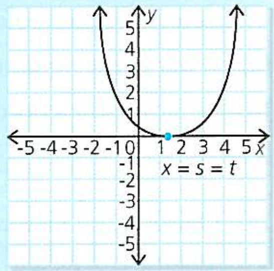


- When a quadratic relation is in factored form, the values of x that are the solutions to $0 = a(x - s)(x - t)$ are called the **zeros** of the quadratic relation. These values correspond to the x-intercepts of the graph of the relation. The zeros can be determined by setting each factor equal to 0 and solving the resulting equation for x. In $y = a(x - s)(x - t)$, the zeros are $x = s$ and $x = t$.

This quadratic relation has two zeros.



This quadratic relation has one zero.



- If $s \neq t$, then the relation has two distinct zeros. If $s = t$, the relation has only one zero at $x = s = t$.
- If the zeros of a quadratic relation are s and t, then the x-coordinate of the vertex is $\frac{s+t}{2}$.
- When the zeros of a quadratic relation are known, the value of a can be determined if some other point (x_1, y_1) on the graph is known. Substitute the coordinates of the point in the factored form, giving $y_1 = a(x_1 - s)(x_1 - t)$, then solve for a.

Example 1

Consider the relation $y = (x + 3)(x - 4)$.

(a) Without graphing, show that the relation is quadratic.

(b) Without using graphing technology, sketch the graph, showing the zeros and vertex.

Solution

(a) Use the relation to set up a table of values and compute the first and second differences.

x	−3	−2	−1	0	1	2	3	4
$y = (x + 3)(x - 4)$	0	−6	−10	−12	−12	−10	−6	0
First Difference	−6	−4	−2	0	2	4	6	
Second Difference	2	2	2	2	2	2		

Since the second differences are constant, the relation is quadratic.

(b) The second differences are positive. Therefore, the parabola opens up.

The zeros of the quadratic relation are the solutions to the equation $0 = (x + 3)(x - 4)$. The values are $x = -3$ and $x = 4$.

The x-coordinate of the vertex is $\frac{-3 + 4}{2} = \frac{1}{2}$.

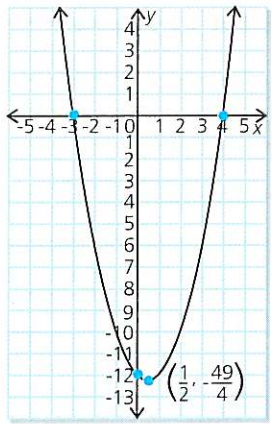
Substitute this x-value in $y = (x + 3)(x - 4)$:

$$y = \left(\frac{1}{2} + 3\right)\left(\frac{1}{2} - 4\right)$$

$$y = \left(\frac{7}{2}\right)\left(\frac{-7}{2}\right)$$

$$y = -\frac{49}{4}$$

The coordinates of the vertex are $\left(\frac{1}{2}, -\frac{49}{4}\right)$.



Example 2

The zeros of a parabola are -3 and 5. The parabola crosses the y-axis at -75.

(a) What is the equation of the quadratic relation?

(b) What are the coordinates of the vertex?

Solution

(a) Develop the factored form of the quadratic relation, $y = a(x - s)(x - t)$. Since the zeros are known, -3 and 5, substitute their values to obtain

$$y = a(x + 3)(x - 5)$$

The parabola crosses the y-axis at -75. This occurs when $x = 0$.

Substitute point $(0, -75)$ in the equation above:

$$-75 = a(0 + 3)(0 - 5)$$

$$-75 = a(3)(-5)$$

$$-75 = -15a$$

$$\frac{-75}{-15} = \frac{-15a}{-15}$$

$$5 = a$$

Substitute $a = 5$ to complete the expression:

$$y = 5(x + 3)(x - 5)$$

(b) Since the zeros are -3 and 5, the vertex has x-coordinate $\frac{-3 + 5}{2} = 1$.

Substitute this into the expression found in (a):

$$y = 5(x + 3)(x - 5)$$

$$y = 5(1 + 3)(1 - 5)$$

$$y = -80$$

The vertex is at $(1, -80)$.

Example 3

This data describes the flight of a plastic glider launched from a tower on a hilltop. The height values are negative whenever the glider was below the height of the hilltop.

(a) How tall is the tower?

(b) Find an equation to model the flight of the glider.

(c) Find the lowest point in the glider's flight.

Time (s)	Height (m)
0	9
1	5.5
2	2.5
3	0
4	–2
5	–3.5
6	–4.5
7	–5
8	–5
9	–4.5
10	–3.5

Time (s)	Height (m)
11	–2
12	0
13	2.5
14	5.5
15	9
16	13
17	17.5
18	22.5
19	28
20	34

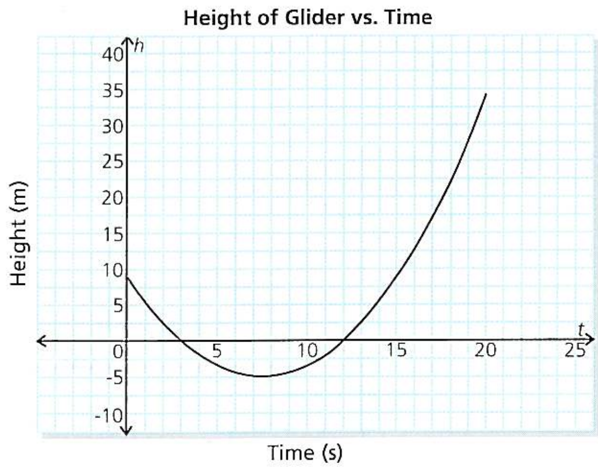
Solution

(a) In this table, a height of 0 represents ground level of the top of the hill. When $t = 0$, the height is 9 m. Therefore, the tower must be 9 m tall.

(b) Graph the flight to see if the graph looks like a parabola. Since it does, locate the zeros and use them to get the equation in factored form.

Let t represent the time in seconds and h the height in metres.

From the table of values, one of the zeros is at $t = 12$ and the other is $t = 3$.



The factored form equation of the parabola must be $h = a(t - 3)(t - 12)$.

Substitute one of the known points, say $(4, -2)$, into this equation to find the value of a.

$$h = a(t - 3)(t - 12)$$
$$-2 = a(4 - 3)(4 - 12)$$
$$-2 = a(1)(-8)$$
$$-2 = -8a$$
$$\frac{-2}{-8} = \frac{-8a}{-8}$$
$$\frac{1}{4} = a$$

The equation of the quadratic model for the flight is $h = \frac{1}{4}(t - 3)(t - 12)$.

(c) The vertex must have a t-coordinate halfway between the zeros, at $t = 7.5$. Substitute $t = 7.5$ into the equation to find the value of h at the vertex.

$$h = \frac{1}{4}(t - 3)(t - 12)$$
$$h = \frac{1}{4}(7.5 - 3)(7.5 - 12)$$
$$h = 0.25(4.5)(-4.5)$$
$$h = -5.0625$$

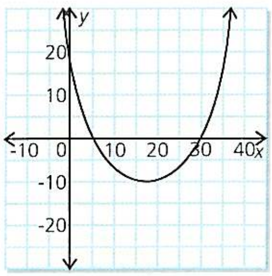
The glider reaches its lowest point, about 5.1 m below the hilltop height, at 7.5 s into the flight.

Practise, Apply, Solve 3.4

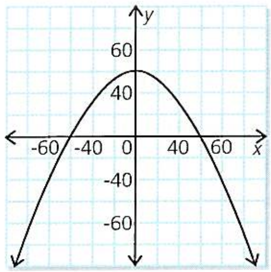
A

1. Examine each parabola. What are the zeros of the quadratic relation?

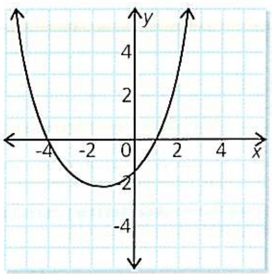
(a)



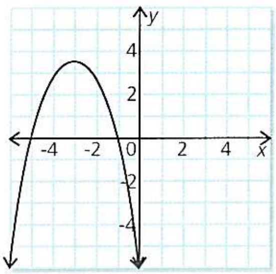
(b)



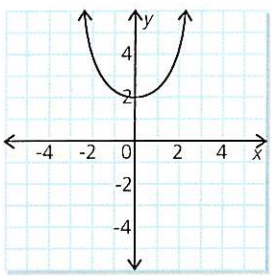
(c)



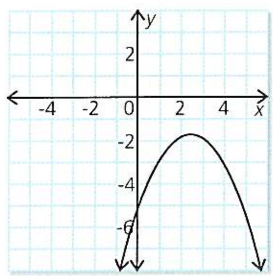
(d)



(e)



(f)



2. Find the equation of the axis of symmetry for each parabola in question 1.

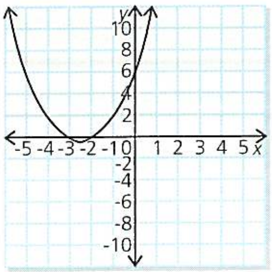
3. Match each factored form equation to the appropriate graph.

(a) $y = (x - 2)(x + 3)$ **(b)** $y = (x - 3)(x + 2)$

(c) $y = (x + 2)(x + 3)$ **(d)** $y = (3 - x)(2 + x)$

(e) $y = (3 + x)(2 - x)$ **(f)** $y = (x - 2)(x - 3)$

i.



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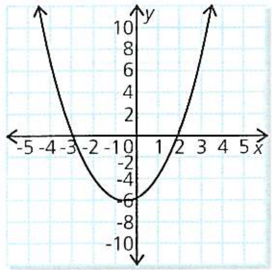
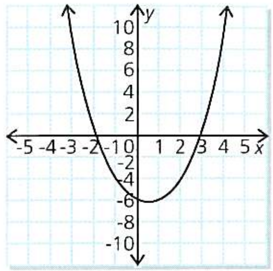


iii.



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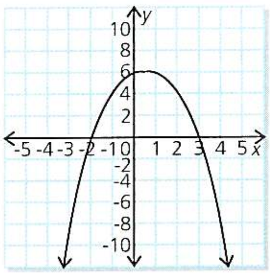
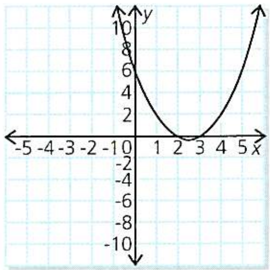
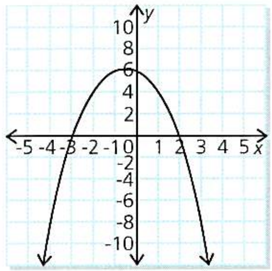


v.



vi.



4. Each quadratic relation has zeros and an optimal value as shown. Sketch the graph. State whether the optimal value is a maximum or a minimum.

	Zeros	Optimal Value
(a)	3 and 7	6
(b)	–6 and –1	–2
(c)	–1 and 7	5
(d)	–9 and 0	–4

5. For each relation, state

i. the x-intercepts

ii. the equation of the axis of symmetry

iii. the coordinates of the vertex

(a) $y = (x + 4)(x + 2)$ **(b)** $y = (x + 5)(2 - x)$

(c) $y = (4 + x)(1 + x)$ **(d)** $y = (1 - x)(3 + x)$

(e) $y = (x - 3)(2 - x)$ **(f)** $y = (x + 1)(x - 4)$

(g) $y = 3(x + 1)(x - 3)$ **(h)** $y = -2(x + 3)(x - 3)$

6. Mirna's Fashion Shoppe is holding a sale. Mirna knows that a decrease in price usually means an increase in sales. These relations are models for expected revenue, based on the selling price x in dollars. Determine the zeros and optimal value.

	Article	Expected Revenue
(a)	sweatshirts	$R = (20 - 2x)(30 + 3x)$
(b)	pants	$R = (30 - 4x)(16 + 2x)$
(c)	socks	$R = (4 - 0.25x)(25 + 2x)$
(d)	suits	$R = (200 - 25x)(9 + 3x)$
(e)	sweaters	$R = (40 - 5x)(20 + 5x)$

B

7. Sketch a graph for each relation. Do not make a table of values or use graphing technology.

(a) $y = (x + 3)(x + 5)$
(b) $y = (x - 3)(x - 5)$
(c) $y = (x - 6)(x - 2)$
(d) $y = -(x - 1)(x - 2)$
(e) $y = 3(x - 5)(x + 1)$
(f) $y = -2(x + 2)(x + 1)$
(g) $y = \frac{1}{2}(x - 4)(x - 2)$
(h) $y = -2(3 - x)(5 - x)$
(i) $y = 10(x - 1)(x + 6)$

8. A quadratic relation has the equation $y = a(x - s)(x - t)$. Find the value of a when

(a) $y = a(x - 2)(x + 6)$ and $(3, 5)$ is a point on the graph
(b) the parabola has zeros of 4 and -2 and a y-intercept of 1
(c) the parabola has x-intercepts of 4 and -2 and a y-intercept of -1
(d) the parabola has zeros of 5 and 0 and a minimum value of -10
(e) the parabola has x-intercepts of 5 and -3 and a maximum value of 6

9. Determine the equation (in factored form) of the quadratic relation and the direction of opening of the parabola.

	x-Intercepts	y-Intercept
(a)	–2 and 4	5
(b)	–2 and 4	–5
(c)	–5 and –2	4
(d)	–5 and –2	–4
(e)	3 and 8	6
(f)	3 and 8	–6

10. Sketch the graphs in question 9. Put any graphs that have the same axis of symmetry on the same axes.

11. Determine the equation of each parabola in question 4.

12. **Knowledge and Understanding:** A parabola has zeros at (5, 0) and (−3, 0) and passes through point (6, 18).

(a) Determine the equation of the axis of symmetry.

(b) Determine the equation of the parabola.

(c) Determine the coordinates of the vertex.

(d) Sketch the graph of the parabola.

13. A ball is thrown upward from the roof of a 25 m building. The ball reaches a height of 45 m above the ground after 2 s and hits the ground 5 s after being thrown.

(a) Use the fact that the relation between the height of a projectile and time is quadratic to draw an accurate graph of the relation on graph paper.

(b) Carefully fold the graph along the axis of symmetry and extend the short side of the parabola to match the long side.

(c) Where does the extended graph cross the time axis?

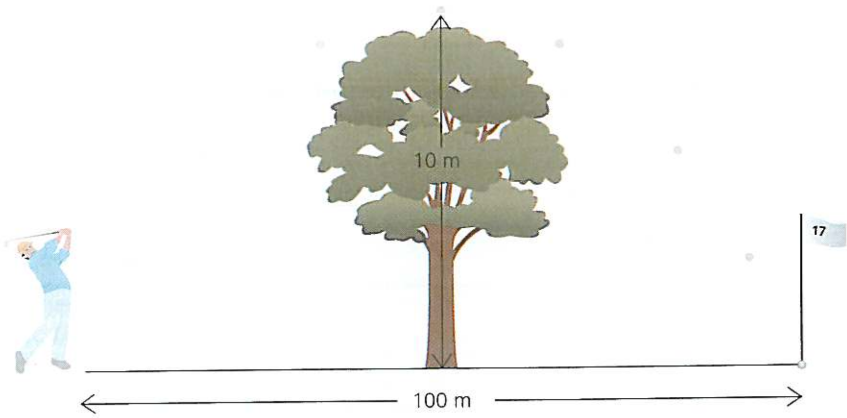
(d) What are the zeros of the relation?

(e) What are the coordinates of the vertex of the parabola?

(f) Determine the algebraic expression that models this situation.

(g) What is the meaning of each zero?

14. **Application:** Angus is playing golf. The diagram (not to scale) shows him making a perfect shot to the pin. Determine the height of the ball when it is 15 m from the hole by using the information in the diagram to determine a quadratic relation for height vs. distance travelled.

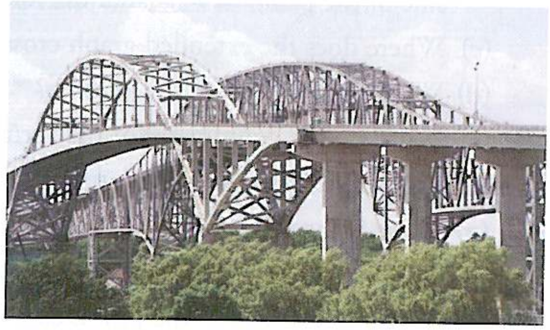


15. A parabolic arch is used to support a bridge. Vertical support columns set in the ground reinforce the arch every 2 m along its length.

This table of values shows the length of the columns in terms of their placement relative to the centre of the arch. Negative values are to the left of the centre point. Write an algebraic model that relates the length of each column to its horizontal placement.

Distance from Centre of Arch (m)	Length of Support Column (m)
−10	70.0
−8	80.8
−6	89.2
−4	95.2
−2	98.8
0	100.0
2	98.8
4	95.2
6	89.2
8	80.8
10	70.0

16. The second span of the Bluewater Bridge, in Sarnia, Ontario, is supported by a pair of steel parabolic arches. The arches are set in concrete foundations that are on opposite sides of the St. Clair River 281 m apart. The top of each arch rises 71 m above the river. Determine the algebraic expression that models the arch.



the Bluewater Bridge

17. This table gives the height of a golf ball at different times during its flight.

(a) Create a scatter plot and draw a graph of best fit.

(b) Use the graph to approximate the zeros of the relation.

(c) Find an algebraic expression that models the flight of the ball.

(d) Use the expression to determine the maximum height of the ball.

Time (s)	Height (m)
0.0	0.000
0.5	10.175
1.0	17.900
1.5	23.175
2.0	26.000
2.5	26.375
3.0	24.300
3.5	19.775
4.0	12.800
4.5	3.375

18. A ball is thrown up into the air from the top of a building. This table gives the height of the ball at different times during its flight.

(a) How tall is the building?

(b) How high is the ball at 2.5 s? 3.5 s?

(c) After many seconds does the ball reach its maximum height?

(d) When does the ball hit the ground?

Time (s)	Height (m)
0	10
1	35
2	50
3	55
4	50
5	35
6	10

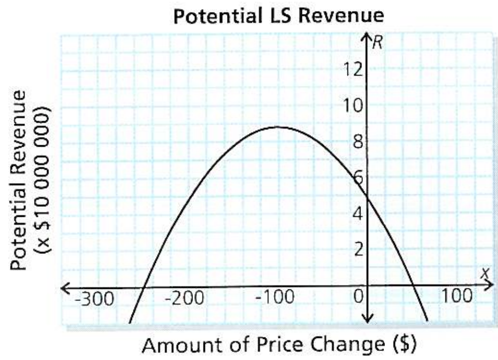
19. Neil sets the prices in the Hardware To Go store. His research shows that an increase of 10¢ in the price of a package of batteries causes a drop in sales of 10 packages per day. The stores normally sell 600 packages of batteries per day, at $4.95 per package.

(a) What is the maximum revenue Neil can expect on the sale of batteries?

(b) How many packages of batteries must be sold to generate the maximum revenue?

(c) What is the optimum pricing strategy in this model?

20. **Communication:** A car manufacturer decides to change the price of its new luxury sedan (LS) model to increase sales. The graph shows the relation between revenue and the amount of the price change, as suggested by the marketing department.



(a) Determine the algebraic model that is a reasonable representation for this graph.

(b) Maximum revenue is not the same as maximum profit. What else should the marketing department consider?

21. Check Your Understanding

(a) Explain how knowing the zeros of a quadratic relation and the coordinates of one additional point allows you to completely determine the algebraic expression for the relation.

(b) What information about the graph can be read directly from the factored form of a quadratic relation?

(c) If the factors of an expression multiply to give zero, how do you know that either factor can equal zero?

C

22. For each factored form equation in column 1, find the corresponding expanded and simplified equation in column 2. How did you decide?

Column 1	Column 2
(a) $y = (2x - 3)(x + 4)$	(1) $y = 12x^2 - 5x - 3$
(b) $y = (3x + 1)(4x - 3)$	(2) $y = -2x^2 - 5x + 12$
(c) $y = (3 - 2x)(4 + x)$	(3) $y = 2x^2 + 11x - 12$
(d) $y = (3 - 4x)(1 + 3x)$	(4) $y = 2x^2 + 5x - 12$
	(5) $y = -12x^2 + 5x + 3$
	(6) $y = 12x^2 + 5x - 3$

23. **Thinking, Inquiry, Problem Solving:** For a school experiment, Marcus had to record the height of a model rocket during its flight. However, during the experiment he discovered that the motion detector he was using had stopped working. Before the detector quit, it collected this data.

Time (s)	0	0.5	1.0	1.5	2.0
Height (m)	1.5	12.525	21.1	27.225	30.9

(a) The trajectory of the rocket is quadratic. Complete the table to the time when the rocket hit the ground.

(b) Determine a expression that models the height of the rocket.

(c) Use your expression to find the rocket's height at 3.8 s.

(d) What is the maximum height of the rocket?

The Chapter Problem—Setting the Best Ticket Price

In this section, you found the zeros and optimal value of a quadratic relation. Apply what you have learned to answer these questions about the chapter problem on page 242.

1. Write a formula for the ticket price, based on the change in price.
2. Write a formula for the attendance, based on the change in ticket price.
3. (a) State the algebraic relation for revenue that models the change in ticket price and the corresponding change in attendance. Is there more than one relation? Explain.

   (b) Determine the zeros of the relation.

   (c) Where does the optimal value occur?

   (d) What is the optimal value?

   (e) Without using a graphing calculator or a table of values, sketch the graph of the relation.

   (f) Compare the graph from (e) to the plots and curves you drew in section 3.1.