Properties of Quadratic Relations 3.2

Part 1: Maximizing the Area Enclosed by a Fence

Wanda and Louise raise puppies. They need a rectangular fenced enclosure for the puppies to run and play in. A contractor said it will cost \$30/m of fence, and they have \$480 to spend.

Think, Do, Discuss

1. (a) What variables in a rectangular design will Wanda and Louise have to consider?
   (b) Draw a sketch that includes all the information you have been given.
   (c) Indicate on your sketch any relationships that exist between the variables.
2. (a) Write an algebraic expression for the area of the fenced enclosure. There should be only one independent variable.
   (b) What is the independent variable in the expression? What is the dependent variable?
   (c) Write the expression in both factored form and expanded form.
3. (a) Use your model to find the dimensions of a rectangular enclosure that will provide the maximum area for the puppies and meet the budget limitations. Use a series of sketches of rectangular designs, a table of values, and a graph to demonstrate that the dimensions you have found are the best for this solution.
   (b) Use first and second differences to decide what type of relation the algebraic expression represents.
4. Refer to the graph and your table of values from step 3. Locate the point that represents the dimensions that give the maximum area.
   (a) Find two points on the curve that are the same distance to the right and to the left of this point. Compare their y-coordinates.
   (b) Repeat (a) for several similar pairs of points.
   (c) Draw a vertical line through the highest point of the curve. Compare the shape of the left part of the curve to the right part of the curve.
   (d) What observation can you make about the shape of the graph of this relation?
   (e) Why are there two points on the graph that represent an enclosed area of $0\ m^2$?
   (f) Explain how these two points relate to the factors of the expression in 2(a).

Part 2: Minimizing Costs

Wanda and Louise purchased the fencing material described in part 1, but then they decided to build two separate square enclosures instead of one enclosure. They want to be able to separate the adult dogs from the puppies.

The ground inside each enclosure has to be covered with sod. They want to use all the fencing they purchased, and to minimize the amount of sod they need to buy.

Think, Do, Discuss

1. **(a)** What variables will Wanda and Louise have to consider?
(b) Draw a sketch that shows all the information available to you.
(c) Indicate on your sketch any relationships that exist between the variables.

2. **(a)** Develop an algebraic expression with one variable to model the total area of the fenced enclosures.
(b) Use this model to find the dimensions of the two enclosures that will cover the minimum ground area and still use all the available fencing material. Use a table of values and a graph to demonstrate that the dimensions you have found are the best for this situation.
(c) Use first and second differences to determine the type of relation represented by the algebraic expression.

3. Refer to the graph and the table of values you created in (2). Locate the point that represents the dimensions that cover the minimum area.
(a) Find two points that are the same distance to the right and to the left of this optimal point. How do their y-coordinates compare?
(b) Repeat (a) for several similar pairs of points.
(c) What can you say about the shape of the graph of this relation?
(d) The relation in part 1 had points that represented an enclosed area of 0 m^2. Explain why this graph has no such points.

4. Examine the graphs and the tables of values for parts 1 and 2.
(a) What relation do you see between the second differences and which way the graphs open?
(b) Examine the graphs and the tables of values for the quadratic relations in section 3.1. Do they confirm the hypothesis you made in (a) or not?

Key Ideas

- The graph of a quadratic relation is called a **parabola.**
- The **vertex** of a parabola is the point on the graph with the greatest y-coordinate if the graph opens down or the least y-coordinate if the graph opens up.
- When a quadratic relation is used to model a situation, the y-coordinate of the vertex corresponds to an **optimal value.** Depending on the direction of opening of the parabola, this represents either a **maximum** or a **minimum** value of the quantity being modelled by the dependent variable. The maximum or minimum value is always associated with the vertex.
- The direction of opening of the parabola can be determined from the sign of the second differences in the table of values of the quadratic relation.
  - ◆ If the constant value of the second differences is positive, then the parabola opens up (figure 1).
  - ◆ If the constant value of the second differences is negative, then the parabola opens down (figure 2).
- A parabola is symmetrical with respect to a vertical line through its vertex. The line is called the **axis of symmetry** of the parabola and the vertex lies on the axis of symmetry. If the coordinates of the vertex are (h, k), then the equation of the axis of symmetry is $x = h$ (figure 3).
- The axis of symmetry is the perpendicular bisector of the segment joining any two points on the parabola that have the same y-coordinates. If the parabola crosses the x-axis, the x-coordinates of these points are called the **zeros**, or **x-intercepts** of the relation, and the vertex is directly above or below the midpoint of the segment joining the zeros (figure 4).

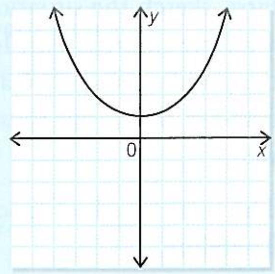


figure 1

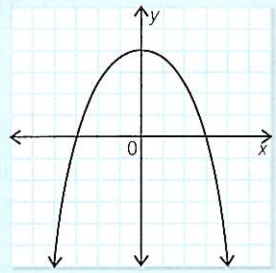


figure 2

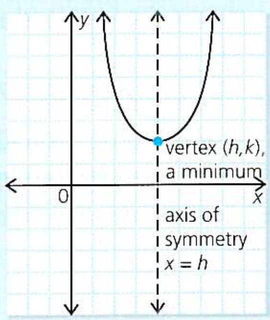


figure 3

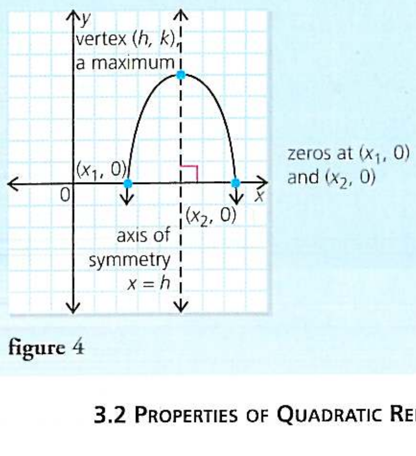
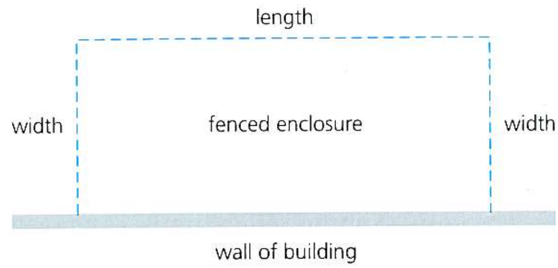


figure 4

Example 1

Phil wants to make the largest possible rectangular vegetable garden using 18 m of fencing. The garden is right behind the back of his house, so he has to fence it on only three sides. Determine the dimensions that maximize the area of the garden.

Solution



First, sketch the situation, showing all the available information. Identify the variables in the problem and state any possible relationships between them. Put these on the diagram as well.

The total length of fencing to be used is 18 m. Therefore,

$$\text{length} + 2(\text{width}) = 18$$

or

$$L + 2W = 18$$

Solving for L,

$$L = 18 - 2W$$

The area of the enclosure is

$$A = LW \qquad \text{Substitute } L = 18 - 2W.$$

$$A = (18 - 2W)W$$

Use this algebraic model to produce a table of values, then draw the graph. Figure 1 shows the results of graphing the quadratic relation on a TI-83 Plus calculator.

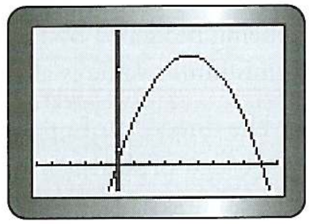


figure 1

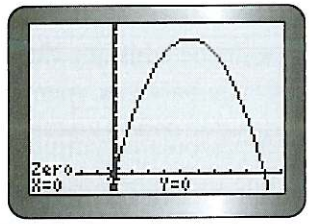


figure 2

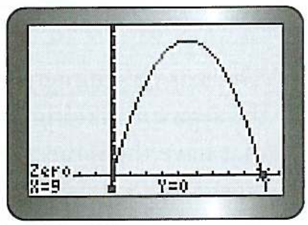


figure 3

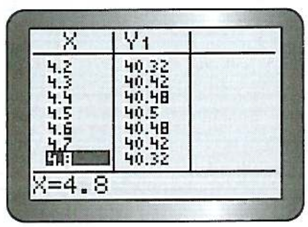


figure 4

From the graph, the zeros are at $x = 0$ and $x = 9$ (figures 2 and 3). The x-coordinate of the vertex is halfway between the zeros, at $x = 4.5$. This is confirmed by looking at the graph or the table of values produced by the calculator when you press [2nd] [GRAPH] (figure 4).

The maximum area enclosed by the fence is 40.5 m^2, when the width, W, is 4.5 m. The length is 9 m.

Example 2

Consider the quadratic relation $y = x(6x - 18)$.
Without making a table of values or drawing the graph, give

(a) the coordinates of the vertex of the parabola
(b) the direction of opening
(c) the axis of symmetry

Solution

(a) The zeros of the parabola are found where it crosses the *x*-axis, at the points where $y = 0$. Substitute $y = 0$ in the relation $y = x(6x - 18)$.

$$0 = x(6x - 18)$$

$$x = 0 \text{ or } 6x - 18 = 0$$

The zeros of the parabola are at $x = 0$ and $6x = 18$, or $x = 3$.

The *x*-coordinate of the vertex is halfway between the zeros. So, the vertex must have $x = \frac{0 + 3}{2}$ or $\frac{3}{2}$. To find the *y*-coordinate of the vertex, substitute $x = \frac{3}{2}$ in the relation.

$$y = x(6x - 18)$$

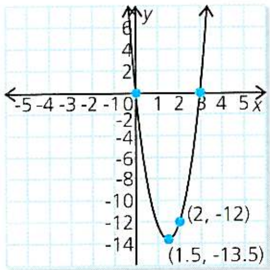
$$y = \frac{3}{2}\left(6\left(\frac{3}{2}\right) - 18\right)$$

$$y = \frac{3}{2}(9 - 18)$$

$$y = \frac{-27}{2} \text{ or } -13.5$$

The vertex of the parabola is at $\left(\frac{3}{2}, -\frac{27}{2}\right)$ or $(1.5, -13.5)$.

(b) Direction of Opening



Since the *x*-coordinate of the vertex is known, you can substitute a value for *x* on either side of the vertex and determine whether the vertex is lower or higher than that point.

For example, substituting $x = 2$ in $y = x(6x - 18)$ gives $y = 2(6(2) - 18) = -12$. The *y*-coordinate of the vertex is $-\frac{27}{2}$ or -13.5, which is less. That means the vertex is a minimum and the graph opens up.

(c) The *x*-coordinate of the vertex is $\frac{3}{2}$. Therefore, the equation of the axis of symmetry is $x = \frac{3}{2}$.

Did You Know?

Before electronic calculators came into common use, people used other tools to perform complex mathematical operations, such as the slide rule and the abacus.

The abacus was invented in China centuries ago, and continues to be used widely today. In fact, expert abacus users often calculate long, involved answers faster than people who use electronic calculators.

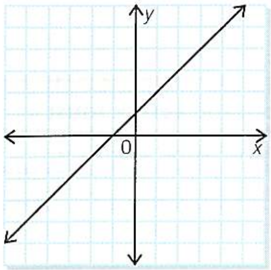
Do some research. How do slide rules and abacuses work? Why are decimals important in the operation of these calculators?

Practise, Apply, Solve 3.2

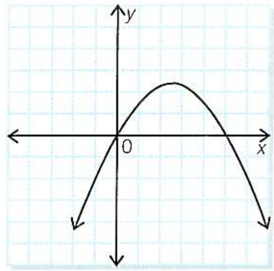
A

1. Which graphs appear to represent quadratic relations?

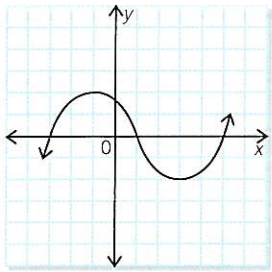
(a)



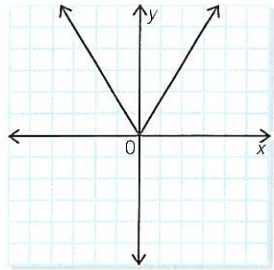
(b)



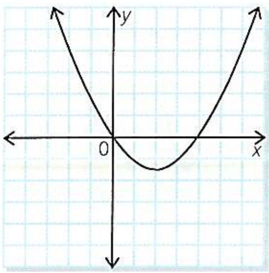
(c)



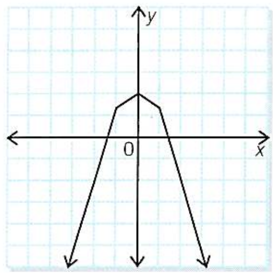
(d)



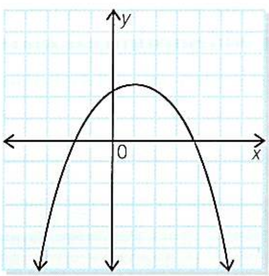
(e)



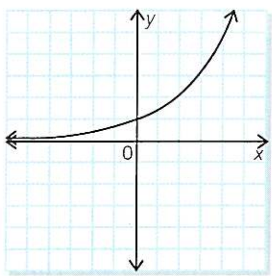
(f)



(g)



(h)



2. Each table of values represents data of a quadratic relation. Decide, without graphing, whether the parabola opens up or down.

(a)

x	−3	−2	−1	0
y	2.5	5	6.5	7

(b)

x	−2	−1	0	1	2
y	0	−5	0	15	40

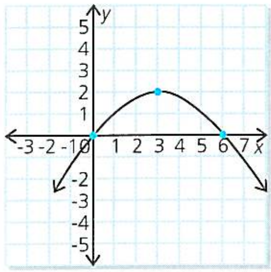
(c)

x	−2	−1	0	1	2
y	−3	3	5	3	−3

(d)

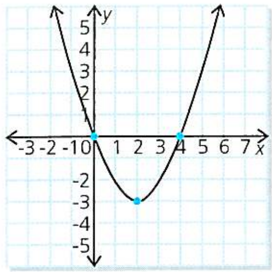
x	0	1	2	3	4
y	−1	4	15	32	55

3. Examine this parabola.



(a) What are the coordinates of the vertex?
(b) What is the optimum value?
(c) What is the equation of the axis of symmetry?
(d) What are the zeros of the relation?
(e) If you calculated the second differences, what would their sign be? Explain.

4. Examine this parabola.



(a) What are the coordinates of the vertex?
(b) State the optimum value. Is it a minimum or a maximum?
(c) What is the equation of the axis of symmetry?
(d) What are the zeros of the relation?
(e) If you calculated the second differences, what would their sign be? Explain.

5. The x-intercepts of a quadratic relation are 0 and 5. The second differences are positive.
(a) Explain whether the optimum value is a maximum or a minimum.
(b) What value of the independent variable produces the optimum value?
(c) Is the optimum value a positive or negative number? Explain.

6. The zeros of a quadratic relation are 0 and 15, and the second differences are negative.
(a) Explain how you can tell whether the optimum value is a maximum or minimum.
(b) What value of the independent variable produces the optimum value?
(c) Is the optimum value positive or negative?

7. Determine the zeros of each quadratic relation.
(a) $A = w(20 - w)$ **(b)** $A = L(12 - L)$ **(c)** $y = x(3x - 6)$
(d) $A = 2w(18 - w)$ **(e)** $y = 2x(5x + 6)$ **(f)** $y = x(16 - 3x)$

8. Two parabolas have the same x-intercepts, 0 and 10. One has a maximum value of 2. The other has a minimum value of -4. Sketch the graphs on the same axes.

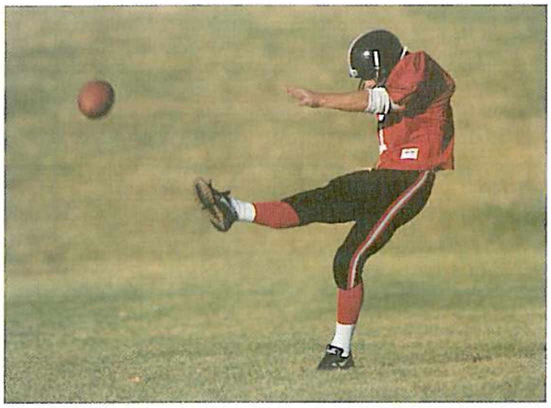
B

9. For each relation
i. write it in factored form by removing a common factor
ii. determine the zeros
iii. state the equation of the axis of symmetry of the parabola
(a) $A = 15w - w^2$ **(b)** $A = 24L - L^2$
(c) $y = 2x^2 - 10x$ **(d)** $y = 15x - 2x^2$

10. Each pair of points (x, y) is located on opposite sides of the same parabola. Determine the equation of the axis of symmetry of the parabola.

(a) $(3, 2), (9, 2)$
(b) $(-18, 3), (7, 3)$
(c) $(-2, -5), (-5, -5)$
(d) $(6.5, -4), (9.0, -4)$
(e) $(-5.25, -2.5), (3.75, -2.5)$
(f) $(-4\frac{1}{2}, 5), (-1\frac{1}{2}, 5)$
(g) $(-3\frac{1}{8}, -2), (7\frac{3}{8}, -2)$
(h) $(s, 0), (t, 0)$

11. **Knowledge and Understanding:** A football is kicked straight up in the air. Its height above the ground is approximated by the relation $h = 25t - 5t^2$, where h is the height in metres and t is the time in seconds.



(a) What are the zeros of the relation? When does the football hit the ground?
(b) What are the coordinates of the vertex?
(c) Use the information you have found to graph the relation.
(d) What is the maximum height reached by the football? After how many seconds does that occur?

12. The zeros of a quadratic relation are 0 and 6. The relation has a minimum value of -9.

(a) Sketch the graph of the parabola that satisfies these conditions.
(b) Find the equation of the parabola.

13. **Communication:** The underside of a bridge is a parabolic arch defined by the equation $y = -0.1x^2 + 2x$ with x and y measured in metres. Explain how to sketch the graph of the structure without using a table of values or graphing technology.

14. Two quadratic relations have the same x-intercepts, 0 and 8. One has a minimum value of -10. The other has a maximum value of 10.

(a) Sketch both relations on the same axes.
(b) Determine the equation of each relation. Write the expressions as polynomials of degree 2.
(c) Compare the coefficients of the x^2-terms. How do the coefficients relate to the direction of opening of the parabola?

15. Two parabolas are defined by $y = 10x - x^2$ and $y = 3x^2 - 30x$ respectively, with both x and y measured in metres. What is the difference between their optimum values?

16. **Application:** Wanda and Louise consider using 30 m, 50 m, or 70 m of fencing to build their puppy run. They want it to be rectangular, using one wall of the kennel instead of fencing on one side. Set up the required relations to model their options and determine the maximum area for each option.

17. **Check Your Understanding**

(a) Describe how the symmetry of a parabola allows you to find the coordinates of the vertex if you know the zeros.

(b) Explain how the second differences can be used to determine if a quadratic relation has a maximum or a minimum value.

(c) Explain how the zeros of a parabola can be determined from the factored form of the quadratic relation.

(d) Describe the relation between the coordinates of the vertex and the optimal value of a quadratic relation.

C

18. **Thinking, Inquiry, Problem Solving**
Canada Post will deliver parcels only if they are less than a certain maximum size: the combined length and girth cannot exceed 297 cm. (*Girth* is the total distance around the cross-section of the parcel.)


girth = distance around the box

length of the box

Canada Post delivers a crate with the largest possible surface area to your house. What is the surface area of the crate in square metres?

The Chapter Problem—Setting the Best Ticket Price

In this section you used quadratic relations to maximize different situations. Apply what you learned to answer these questions about the chapter problem on page 242.

1. For each table of values that you found when you worked on the chapter problem in section 3.1, calculate the first and second differences. What type of relation does the data represent?
2. Extend the graphs using symmetry to determine where each graph meets the horizontal axis.
3. Using the graph, determine the maximum revenue and the change in price that corresponds to it.