3.1 Quadratic Relations

Part 1: Linear and Nonlinear Models

Examine each situation and think about the kind of mathematical model that would apply.

Blood Alcohol Content

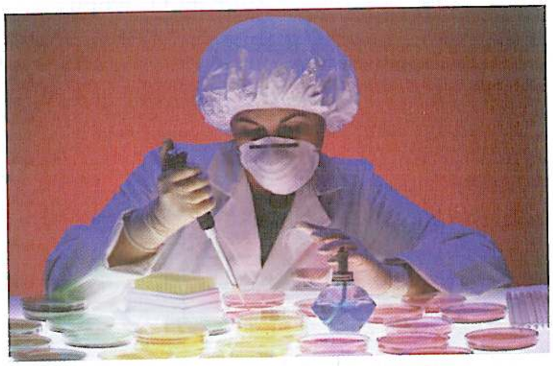
Drinking and driving is one of the major causes of vehicle accidents. A driver's *blood alcohol content* (BAC) depends on body mass and the amount of alcohol consumed. The table shows the number of standard-size drinks that produce a BAC of 0.05% within a 2 h period.

(Note: One standard-size drink is 30 mL (one ounce) of 80-proof spirits, 90 mL (3 ounces) of wine, or 240 mL (8 ounces) of 5% beer. Also, although the figures shown here are accurate, individual results would vary.)

Alcohol Consumption to Reach 0.05% BAC in 2 Hours

Body Mass (kg)	46	55	64	73	82	91	100	109
Number of Drinks	2.5	2.9	3.3	3.7	4.1	4.5	4.9	5.3

Bacteria Growth



The population of a bacteria colony is measured over a 6 h period, resulting in this data.

Population of Bacteria

Time (h)	0	1	2	3	4	5	6
Bacteria Count	1900	3900	8000	15 900	31 900	64 100	127 900

Falling Water

A garden hose sprays a stream of water across a lawn. The table records the approximate height of the stream above the lawn at various distances from the nozzle.

Water Height vs. Distance from Nozzle

Distance from Nozzle (m)	0	1	2	3	4	5	6	7	8
Height Above Lawn (m)	0	0.9	1.6	2.1	2.4	2.5	2.4	2.0	1.4

Think, Do, Discuss

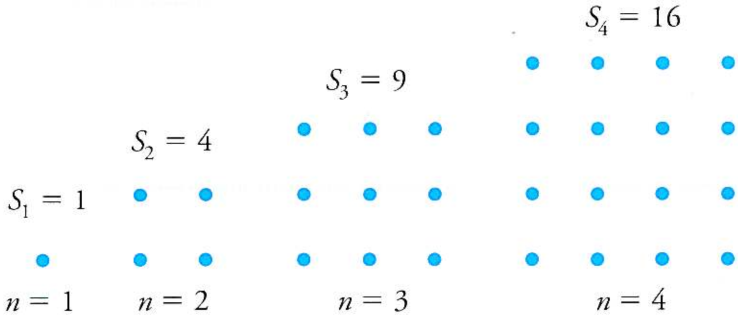
1. (a) Draw and label a separate scatter plot for each situation.
   (b) Which situations can be modelled with a linear relation? Justify your answer. Draw the line of best fit for these situations and comment on how well the model fits the data.
   (c) For those situations that require a nonlinear model, use a piece of string to make a curve that matches the pattern of the data points. Some data points may not fit the pattern exactly. Describe how to place the string when this happens.
   (d) Draw the curve of best fit for the nonlinear models by following the shape of the string. How well does the curve fit the data?
2. Use your graphs to answer these questions.
   (a) For an adult with a body mass of 60 kg, how many drinks of alcohol would result in a blood alcohol content of 0.05% after 2 h?
   (b) How many bacteria would there be after 7 h?
   (c) What is the maximum height of the water stream? How far from the nozzle does the water reach this height?
3. (a) The differences between consecutive values of the dependent variable are called **finite differences**, or **first differences**. Calculate the first (or finite) differences for each table of values.
   (b) Explain how the values in the difference tables support your decision about linear and nonlinear models in question 1.
   (c) The value of a first difference represents the amount of change in the dependent variable (Δy) caused by a change in the independent variable (Δx). The ratio of these changes, $\frac{\Delta y}{\Delta x}$, can be used to estimate the rate of change of one quantity relative to the other quantity. For each situation, describe which quantities are changing and what the rate of change means.
   (d) For which of the situations is the rate of change constant? What does this tell you about the relation being modelled?
4. (a) For which of the situations is the rate of change variable? What does this tell you about the relation being modelled?
   (b) How does a variable rate of change affect the graph?

Part 2: Number Patterns

Square Number Patterns

Ancient mathematicians were fascinated by **figurate numbers**: sequences that represent the number of markers needed to construct common geometric shapes. For example, the **square numbers** are 1, 4, 9, 16, and so on.

The first square number (S_1) is 1; the second square number (S_2) is 4, and so on.



Complete this table of square numbers.

n	1	2	3	4	5	6	7	8
S_n	1	4	9	16				

Summing Patterns

The great mathematician Karl Friedrich Gauss showed his talents even as a young boy. One day, his teacher, who hoped to keep the class busy, instructed the students to add the whole numbers from 1 to 100. Gauss gave the correct answer almost immediately.

Karl Friedrich Gauss (1777–1855)

Gauss's reasoning was as follows:

Write out the sequence of numbers (or imagine writing them out). Below the first sequence, write the sequence again, in reverse order.

1	2	3	4	...	97	98	99	100
100	99	98	97	...	4	3	2	1

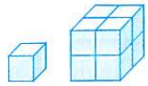
Gauss realized that each vertical pair adds to 101 and that there are exactly 100 pairs. The total, for the two copies of the sequence, is (100)(101) = 10 100. For one copy of the sequence, the total is half as much.

$$S_{100} = \frac{(100)(101)}{2} = 5050$$

Use Gauss's method to complete this table for values of n from 1 to 20.

n	1	2	3	4	5	6	7	...	20
Sum of numbers from 1 to n	1	3	6						

Surface Area Patterns



The surface area of a 1 cm cube is 6 cm^2, because the cube has 6 square faces that are each 1 cm × 1 cm. For a 2 cm cube, the 6 faces are 2 cm × 2 cm squares and the surface area is 24 cm^2.

Copy and complete this table.

Cube Side Length (cm)	1	2	3	4	5	6	7	...	10
Surface Area (cm^2)	6	24							

Think, Do, Discuss

1. Refer back to the section Square Number Patterns.

(a) Write a formula for the nth square number in terms of n.

(b) Predict the value of the 12th square number. If there was a square number between S_5 and S_6, estimate its value. Explain why such a number is not really possible.

(c) Draw a scatter plot of the data.

2. Refer back to the section Summing Patterns.

(a) Write a formula for the sum of the first n whole numbers in terms of n.

(b) Draw a scatter plot of the data.

3. Refer back to the section Surface Area Patterns.

(a) Write a formula for the surface area of a cube in terms of its side length l.

(b) Draw a scatter plot of the data.

4. **(a)** How are the formulas in questions 1, 2, and 3 similar?

(b) How are the graphs the same?

(c) How are they different?

5. **(a)** Compute the first differences for each table of values.

(b) Explain how the values of the differences show that the relationships are not linear.

(c) How is the pattern in the differences the same among these relationships?

(d) How is the pattern in the differences different?

6. **(a)** Refer to the sequence of first differences that you found in question 5. Calculate the **second differences**, that is, the differences between consecutive first differences.

(b) What do you notice about the second differences for all three relationships?

7. Which situations in part 1 have graphs that are similar to those in part 2? What type of mathematical model would best fit those situations? Explain.

8. (a) Calculate the second differences for the relations in part 1.
   (b) Which relations in part 1 have patterns in their second differences similar to those in part 2?
   (c) What does that suggest about the type of mathematical model that would best fit those situations?
   (d) In the formula for the dependent variable, what is the value of the highest exponent when the first differences are constant?
   (e) What is the highest exponent when the second differences are constant?

Focus 3.1

Key Ideas

- Sometimes a **curve of best fit** is a more appropriate model for data than a line of best fit. This is true when the data points seem to fit a recognizable pattern that is not a straight line. In such a case, try to draw a smooth curve that passes through as many of the data points as possible. Visualize where the curve should lie between the actual plotted data points. A piece of string may help you to decide the shape and location of the curve.
- The values of the **first differences** in a table of values determine if the relation is linear.
  - For constant increments of the independent variable, a relation is linear if the first differences of the dependent variable are constant.

    For example, the first differences in this table of values are constant, so the relation is linear.

x	0	1	2	3	4
y	2	3	4	5	6
First Difference	1	1	1	1	

  - For constant increments of the independent variable, a relation is **quadratic** if the second differences of the dependent variable are constant.

    For example, the second differences in this table of values are constant, so the relation is quadratic.

x	0	1	2	3	4
y	2	3	6	11	18
First Difference	1	3	5	7	
Second Difference	2	2	2		

- A linear relation models a phenomenon where the rate of change is constant. A nonlinear relation models a phenomenon with a variable rate of change.
- The **degree** of a one-variable polynomial is the highest exponent that appears in any term of the expanded form of the polynomial.
- A polynomial of degree 2 models a quadratic relation.

Example 1

A car driver puts on the brakes and skids through an intersection. The investigating police officer knows that the distance a car skids depends on the speed of the car just before the brakes are applied. She uses a chart to determine the car's speed before the skid.

Speed (km/h)	0	10	20	30	40	50	60	70	80	90	100
Length of Skid (m)	0	0.7	2.8	6.4	11.4	17.8	25.7	35.0	45.7	57.8	71.4

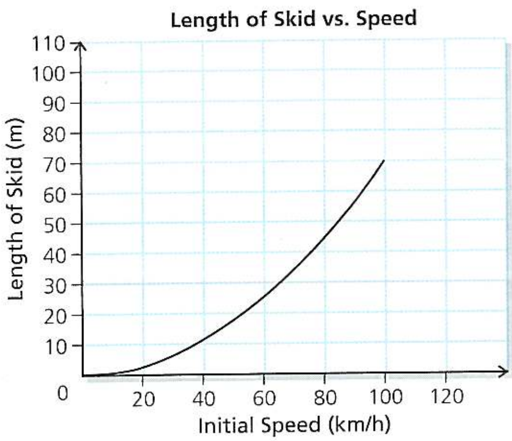
(a) Draw the curve of best fit for this data.

(b) Use the curve to estimate the initial speed of the car if the skid mark is 104 m long.

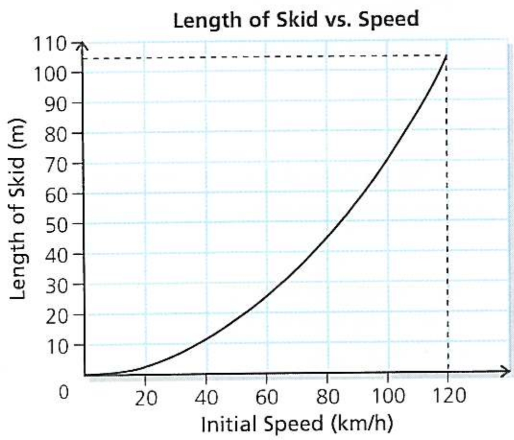
(c) Determine if either a linear or a quadratic relation can be used to model the data.

Solution

(a)



(b)



If the car skids 104 m, its initial speed was about 120 km/h.

(c) Compute the first and second differences.

Speed (km/h)	0	10	20	30	40	50	60	70	80	90	100
Length of Skid (m)	0	0.7	2.8	6.4	11.4	17.8	25.7	35.0	45.7	57.8	71.4
First Difference	0.7	2.1	3.6	5.0	6.4	7.9	9.3	10.7	12.1	13.6	
Second Difference	1.4	1.5	1.4	1.4	1.5	1.4	1.4	1.4	1.5		

The first differences are not the same: the relation is definitely not linear. The second differences are almost the same, so a quadratic model should be suitable for this relation.

Example 2

Two parachutists jump out of a plane at 2000 m. The first opens his parachute almost immediately. The other free-falls for the first 10 s. The table records their height above the ground at different times. Show that the first parachutist has a constant rate of descent and the second has a variable rate of descent during the period of measurement.

Time (s)	0	2	4	6	8	10
Height of Jumper 1 (m)	2000	1980	1960	1940	1920	1900
Height of Jumper 2 (m)	2000	1980	1920	1820	1680	1500

Solution

To determine the rate of descent, examine how the distance each jumper falls during each 2 s interval changes. This is the first difference in the table of values.

Time (s)	0	2	4	6	8	10
Height of Jumper 1 (m)	2000	1980	1960	1940	1920	1900
First Difference	–20	–20	–20	–20	–20	

The first jumper has a constant rate of descent of 20 m in each 2 s interval. This could be modelled by a linear relation with a slope of $\frac{-20}{2} = -10$. The rate of descent is -10 m/s.

Time (s)	0	2	4	6	8	10
Height of Jumper 2 (m)	2000	1980	1920	1820	1680	1500
First Difference	–20	–60	–100	–140	–180	
Second Difference	–40	–40	–40	–40		

The second jumper has a variable rate of descent. During each 2 s interval, the change in height is different. Since the second differences are constant, a quadratic relation could be used to model the situation.

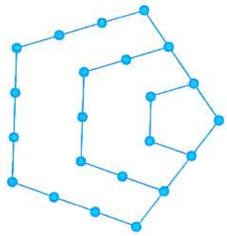
Example 3

Pentagonal numbers are another group of figurate numbers. They are generated in much the same way as square numbers, but are based on pentagons rather than squares.

Determine the kind of relation that would best model the pattern of pentagonal numbers.

Solution

Draw the diagram for the first four pentagonal numbers.



The diagram shows that

$P_1 = 1$

$P_2 = 5$

$P_3 = 12$

$P_4 = 22$

Extend the pattern for P_5 and P_6. Construct a table of values showing the first and second differences.

n	1	2	3	4	5	6
P_n	1	5	12	22	35	51
First Difference	4	7	10	13	16	
Second Difference	3	3	3	3		

Since the second differences are constant, the model should be a quadratic relation.

James Wesley Graham (1932-1999)

James Wesley Graham

Today the University of Waterloo has a world-renowned reputation for the excellence of its graduates in Computer Science. Much of that fame is based on the pioneering work of Wes Graham and his colleagues. When the university began in 1957, computers were hard to program and only a few students could use them. To make computers easier to program for students, Professor Graham helped to create WATFOR (WATerloo FORtran), an educational version of the programming language FORTRAN. Previous implementations of computer languages were difficult to learn because students usually took hours to find and identify errors. WATFOR was simpler to learn and use because students could locate their mistakes more quickly. Thus, the students could try many more examples. WATFOR was just one of Professor Graham's achievements. He was one of the first individuals to connect personal computers in what is now called a "local area network," and to design a portable computer.

Practise, Apply, Solve 3.1

A

1. Examine each pattern. Supply the missing values.

(a) ■, 4, 1, 0, 1, 4, ■

(b) ■, 3, 13, 27, 45, 67, ■

(c) ■, 1, −2, −11, −26, −47, ■

(d) ■, 10, 3, 0, 1, 6, ■

(e) ■, 5, 7, 5, −1, −11, ■

(f) ■, 4, −3, −6, −5, ■

2. i. For each set of data, calculate the first differences and identify the linear and nonlinear relations.

ii. For the nonlinear relations, determine the second differences and identify the quadratic relations.

(a)

x	10	20	30	40
y	21	41	61	81

(b)

x	1	2	3	4
y	4	7	12	17

(c)

x	5	6	8	11
y	−2	−3	−5	−8

(d)

x	0	1	2	3
y	1	−1	−7	−11

(e)

x	0	1	2	3
y	−2	−1	6	25

(f)

x	0	1	2	3	4
y	1	2	4	8	16

3. For the linear relations in question 2, determine the slope.

4. Follow the pattern and determine the missing values.

(a)

y	■	■	■	−2	■	■
First Difference	−3	−3	−3	−3	−3	

(b)

y	■	■	4	■	■
First Difference	■	■	1	■	
Second Difference	2	2	2		

(c)

y	■	−8	■	■	■	■
First Difference	■	6	■	■		
Second Difference	−1	−1	−1			

5. Determine the degree of each polynomial.

(a) $3x - 2$ (b) $\frac{1}{2}r^2$ (c) $2r$

(d) $-4.9t^2$ (e) $x^2 + 3x - 1$ (f) $2x^3 - 3x + x - 4$

(g) $x^2 - 9$ (h) $x(x + 2)$ (i) $x^2(x^3 - 3)$

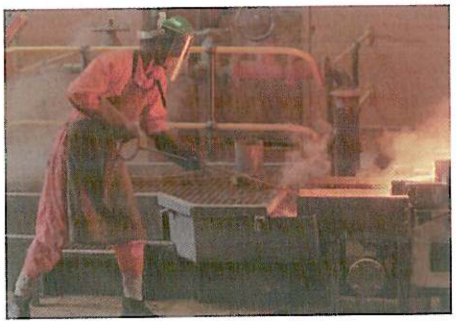
(j) $x(x^2 + 2x + 1)$ (k) $ax + b$ (l) $ax^2 + bx + c$

6. Identify the polynomials in question 5 that could be used to model a quadratic relation.

7. For each set of data, draw a scatter plot. Depending on the scatter plot, draw either a line of best fit or a curve of best fit.

(a) The amount of gold produced for several years is estimated.

Year	1977	1978	1979	1980
Gold Produced (t)	1108	1111	1122	1147



(b) A colony of bacteria is growing in a petri dish. A research scientist estimates the number of bacteria at one minute intervals.

Time (min)	0	1	2	3	4
Bacteria Count (000s)	6	12	23	50	100

(c) Josh dissolves salt in a pot of water. He finds that as the water gets hotter, more salt will dissolve.

Temperature (°C)	0	20	40	60	80	100
Dissolved Salt (g)	52	62	81	103	131	159

(d) Each year thousands of tonnes of garbage are produced. Garbage that is not recycled is dumped in a landfill site or burned. The table shows the approximate amount of waste that has been burned in recent years for a large city.

Year	1994	1995	1996	1997	1998	1999
Waste Burned (t)	86 000	90 000	89 000	95 000	89 000	99 000

B

8. The table represents the average body mass for children up to age 12.

Age (years)	1	2	3	4	5	6	7	8	9	10	11	12
Mass (kg)	11.5	13.7	16.0	20.5	23.0	23.0	30.0	33.0	39.0	38.5	41.0	49.5

(a) Draw a scatter plot.
(b) Draw a best-fit graph.
(c) What type of model best represents the data? Explain.
(d) Describe the goodness of fit of the model, using one of these choices: poor, reasonable, or very good fit. Explain your choice.

9. **Knowledge and Understanding:** A ball is dropped from the roof of a 15-storey building and is timed as it passes various windows. The table shows the results of two repetitions of the experiment.

Height of Ball (storeys)	15	13	11	9	7	5	3	1
Time from 1st Experiment (s)	0	1.1	1.6	2.1	2.3	2.6	2.8	2.9
Time from 2nd Experiment (s)	0	1.0	1.5	2.2	2.3	2.5	2.9	3.0

(a) Draw a scatter plot combining the data from both experiments on the same graph.
(b) Draw a best-fit curve.
(c) Describe the goodness of fit from one of these choices: poor, reasonable, or very good fit. Explain your choice.
(d) Is the data best represented by a linear or nonlinear relation? Explain.

10. A pendulum swings back and forth. The time it takes to make one complete swing and return to the original position is called the **period** of the pendulum. The period changes according to the length of the pendulum.

Length of Pendulum (cm)	6.2	24.8	55.8	99.2	155.0
Period (s)	0.5	1.0	1.5	2.0	2.5

(a) Draw a scatter plot and the curve of best fit.
(b) If a pendulum is 40 cm long, determine its period.
(c) Predict the length of a pendulum if its period is 2.2 s.
(d) Determine the type of model that best describes the relation between the length and the period of a pendulum. Explain.

11. **Communication:** Paymore Shoe Company introduces a new line of neon green high heel running shoes. The table shows the number of pairs sold at one store over an 11-month period.

Month	1	2	3	4	5	6	7	8	9	10	11
Shoes Sold (pairs)	56	60	62	62	60	56	50	42	32	20	6

(a) Draw a scatter plot and the curve of best fit.

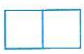
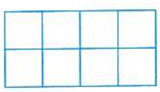
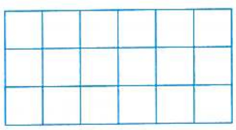
(b) When did the number of shoes sold per month reach its peak?
(c) What has happened by month 11? Explain why this might have occurred.
(d) Determine the rate of change in the number of pairs of shoes sold between months 1 and 2, and months 4 and 5. What does this mean?
(e) Compare the rate of change in shoe sales between months 1 and 2, and months 2 and 3. What does this mean?
(f) Compare the rate of change in shoe sales between months 4 and 5, and months 5 and 6. What does this mean?
(g) Show that the relation between the number of pairs of shoes sold and the time the shoes have been on the market is a quadratic relation. Explain why this model is suitable for the fashion business. How is it unsuitable?

12. A ball is tossed straight up in the air. Its height is recorded every quarter second.

Time (s)	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Height (m)	1.5	3.5	4.9	5.7	5.7	5.2	4.1	2.4	0.1

(a) Draw a scatter plot.
(b) What type of model is a reasonable representation of the relationship between the height of the ball and the time in the air? Explain
(c) Draw the graph that best fits the data.
(d) When does the ball reach its highest point above the ground? What is the ball's height at this point? Be as precise as you can, using your graphical model.
(e) About how long is the ball in the air? Explain.

13. A set of rectangles is made from 1 cm squares. The first rectangle is 1 cm × 2 cm. The next is 2 cm × 4 cm, then 3 cm × 6 cm. The pattern continues with the length always twice the width.
(a) Draw the patterns of squares to help you visualize the pattern. Use your diagrams to complete the table.

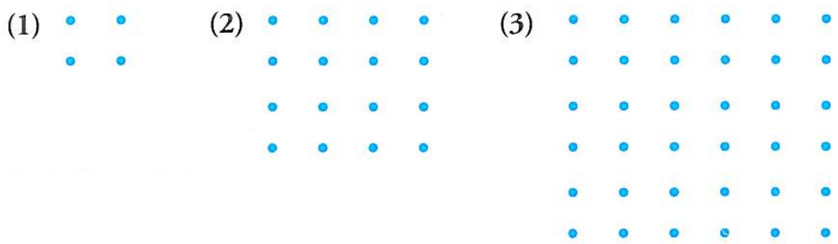
Shape	Width (cm)	Length (cm)	Area (cm^2)
	1	2	2
			
			
next shape in pattern			
next shape in pattern			
next shape in pattern			

(b) Determine the first differences for the area of the rectangles. How do these differences tell you whether the relation between width and area is linear or nonlinear?

(c) Determine the second differences for the area. How does this help determine the type of relation between width and area?

(d) Write an algebraic expression that represents the relation between width and area. What is the degree of the expression?

14. Examine these square dot patterns.



(a) Extend the pattern to form the next two shapes in the sequence.

(b) Complete the table of values in your notebook, where n is the number of each diagram in the pattern and S_n is the total number of dots in each shape.

n	1	2	3	4	5
S_n	4	16	36		

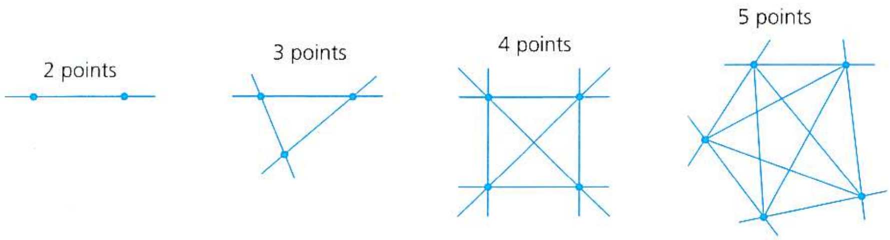
(c) Draw the graph of S_n vs. n.

(d) Does the sequence define a linear or nonlinear relation? Explain.

(e) Determine the second differences. What type of model describes the relationship between the diagram number and the number of dots in each shape? Explain.

(f) Write an algebraic expression that models the relation. What is the degree of the expression?

15. These diagrams show points joined by all possible line segments.



(a) Extend the pattern to include a figure with six points.

(b) Make a table of values that shows the number of points and the number of connecting line segments.

(c) Graph the data.

(d) Does the data represent a linear or nonlinear relation? Explain.

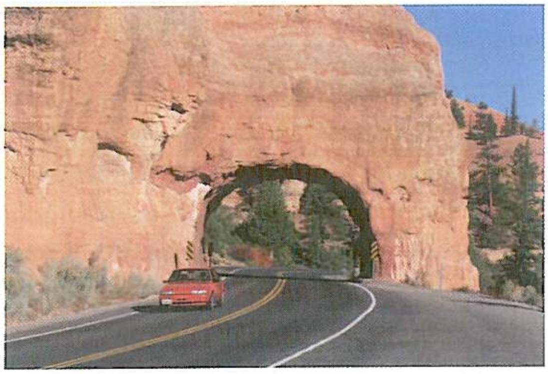
(e) Write an algebraic expression for the number of line segments in terms of the number of points.

(f) Extend the pattern to include a figure with seven points. How many line segments are in this figure?

(g) Use the result from (f) to verify the algebraic expression found in (e).

16. **Application:** When a car is driven, the amount of gas used per kilometre depends on the speed the car is travelling. The table shows an example of this relation.

Speed (km/h)	20	40	60	80	100	120
Cost of Gas (¢/km)	9.1	7.8	7.1	7.1	7.8	9.1



(a) What type of relation seems to be a good model for the data?

(b) Draw the curve of best fit.

(c) What speed is the most cost-efficient?

(d) Explain why lower and higher speeds are not as cost-efficient.

(e) On a 600 km trip, how much money would be saved by driving at the most cost-efficient speed, rather than at 100 km/h? How much longer would the trip take?

17. **Check Your Understanding:** Suppose you are given a table of values representing the results of an experiment. You draw a graph of the data. What kinds of patterns would you look for in the data to decide whether to use a linear model, a quadratic model, or some other nonlinear model to make predictions?

C

18. Find the sum of all the whole numbers from 200 to 399.

19. **Thinking, Inquiry, Problem Solving:** Nina is an O.P.P. officer. One of her duties is to investigate traffic accidents and to prepare accurate reports for use in court. She uses this formula to estimate the speed of a vehicle, based on the length of the skid mark on the road.

$$s = 15.9\sqrt{Df}$$

where

- f is the drag factor. A more precise name is the frictional coefficient of the road surface.
- D is the length of the skid mark (in metres)
- s is the speed when the skid started (in kilometres per hour)

This table gives typical values of f for a variety of road conditions.

Road Surface	Dry	Wet
concrete	0.70	0.40
asphalt	0.65	0.45
gravel	0.50	0.50
ice	0.07	0.05
snow	0.35	0.30

Source: Huntley, I. D, and D. J. G. James. *Mathematical Modeling.* New York: Oxford University Press, 1990.

(a) Create a table of values and a graph showing how the speed estimate varies with skid length for a concrete surface under dry conditions.

(b) Use the graph to estimate the speed of a car at the time the brakes were applied if the skid mark is 50 m long.

(c) If the speed limit for the road is 80 km/h, what is the longest possible skid mark for a car travelling at or below the limit on dry concrete?

(d) If a car is equipped with an anti-lock braking system (ABS), the skid length is typically 10% less. How would that affect your answer in (c)?

(e) Drivers typically take 1.5 s to react to a dangerous situation. How far from an intersection would a driver have to be to safely stop a car travelling at 80 km/h?

(f) Show how the road surface material and moisture conditions affect your answer to (e).

The Chapter Problem—Setting the Best Ticket Price

In this section you used nonlinear models. Apply what you learned to answer these questions about the chapter problem on page 242.

1. Prepare a table of values to show how a decrease in ticket price affects the attendance and the revenue.
2. Prepare another table to show how an increase in ticket price affects attendance and revenue.
3. Draw scatter plots of the data, showing revenue vs. change in price.
4. Draw the curve of best fit for each scatter plot.