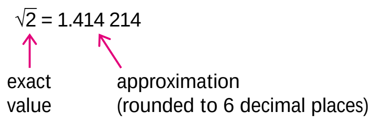
SKILL BUILDER 6.3 Using Special Triangles to Determine Exact Values

Determining the value of a trigonometric function for a given angle can be found to a high degree of accuracy using scientific and graphing calculators. However, these values are close approximations, not exact values. This is due, in large part, to rounding of irrational numbers such as $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$.



In the early stages of trigonometry, triangles were used to generate tables of values for each trigonometric function. A few of these triangles are still in use today to help generate exact values for trigonometry functions involving frequently used angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$.

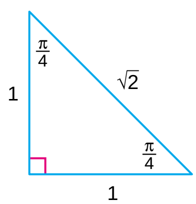
Any right isosceles triangle has angles of $\frac{\pi}{4}$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$. Since the trigonometric ratios are independent of the lengths of the sides, the simplest triangle that contains these angles can be used to generate the trigonometric ratios. This triangle has two equal sides of 1 unit in length.

Calculate the hypotenuse using the Pythagorean theorem.

$$1^2 + 1^2 = x^2$$
$$1 + 1 = x^2$$
$$2 = x^2$$
$$\sqrt{2} = x \qquad \text{Since } x \text{ is a side length it must be positive.}$$

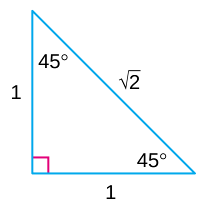
Label the opposite side, the adjacent side, and the hypotenuse. It follows that

In radians,



$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\tan\frac{\pi}{4} = 1$$

In degrees,

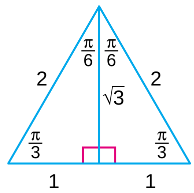


$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$
$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$
$$\tan 45^\circ = 1$$

An equilateral triangle contains three angles of $\frac{\pi}{3}$. Drawing the perpendicular bisector of the base creates two congruent triangles, which both contain angles of $\frac{\pi}{6}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. The simplest case to consider is an equilateral triangle with sides of 2 units. Looking at one of these triangles, the vertical side can be found using the Pythagorean theorem.

$$\begin{aligned} x^2 + 1^2 &= 2^2 \\ x^2 + 1 &= 4 \\ x^2 &= 4 - 1 \\ x^2 &= 3 \\ x &= \sqrt{3} \end{aligned}$$

Since x is a side length, it must be positive.



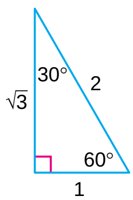
Label the opposite side, the adjacent side, and the hypotenuse. It follows that

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \sin \frac{\pi}{6} = \frac{1}{2}$$

$$\cos \frac{\pi}{3} = \frac{1}{2} \qquad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan \frac{\pi}{3} = \frac{\sqrt{3}}{1} \text{ or } \sqrt{3} \qquad \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

In degrees,



$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \sin 30° = \frac{1}{2}$$

$$\cos 60° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2}$$

$$\tan 60° = \frac{\sqrt{3}}{1} \text{ or } \sqrt{3} \qquad \tan 30° = \frac{1}{\sqrt{3}}$$

Determining an Exact Value

The special triangles can be used to determine the exact value of an expression.

Example 1

Determine the exact value of $\sin \frac{\pi}{4} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \cos \frac{\pi}{6}$.

Solution

Use the special triangles above and substitute the appropriate ratio.

$$\begin{aligned} \sin \frac{\pi}{4} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \cos \frac{\pi}{6} &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{1}{2} + \frac{3}{4} \\ &= \frac{2}{4} + \frac{3}{4} \\ &= \frac{5}{4} \text{ or } 1\tfrac{1}{4} \end{aligned}$$

Special Triangles and Related Angles

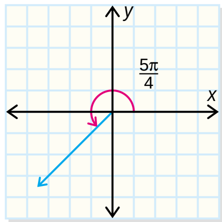
The special triangles can be used to determine the exact value of a trigonometric ratio for angles with $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ as related angles.

Example 2

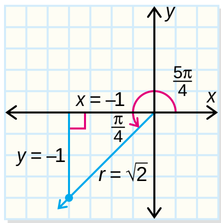
Determine the exact value of $\cos \frac{5\pi}{4}$.

Solution

Sketch the angle in standard position.



Determine the related angle and create a right triangle by drawing a vertical line from the terminal arm, perpendicular to the x-axis. Determine the values of x, y, and r using the $\frac{\pi}{4}$ special triangle and the fact that this angle lies in the third quadrant.



Determine the exact value.

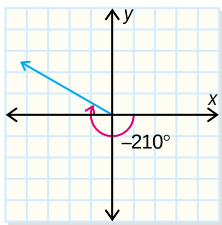
$$\cos \frac{5\pi}{4} = \frac{x}{r}$$
$$= \frac{-1}{\sqrt{2}}$$

Example 3

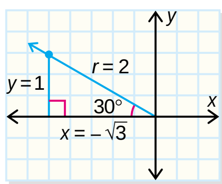
Determine the exact value of $\sin(-210°)$.

Solution

Sketch the angle in standard position.



Determine the related angle and create a right triangle by drawing a vertical line from the terminal arm, perpendicular to the x-axis. Determine the values of x, y, and r using the 30° special triangle and the fact that this angle lies in the second quadrant.



Determine the exact value.

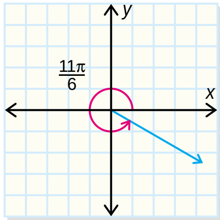
$$\sin(-210°) = \frac{y}{r}$$
$$= \frac{1}{2}$$

Example 4

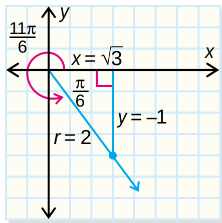
Determine the exact value of $\tan \frac{11\pi}{6}$.

Solution

Sketch the angle in standard position.



Determine the related angle and create a right triangle by drawing a vertical line from the terminal arm, perpendicular to the x-axis. Determine the values of x, y, and r using the $\frac{\pi}{6}$ special triangle and the fact that this angle lies in the fourth quadrant.



Therefore,

$$\tan \frac{11\pi}{6} = \frac{y}{x}$$
$$= \frac{-1}{\sqrt{3}}$$

Using the Special Triangles to Solve Linear Trigonometric Equations

Sometimes special triangles can be used to help determine the solution.

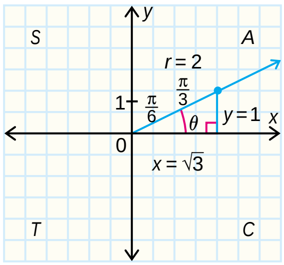
Example 5

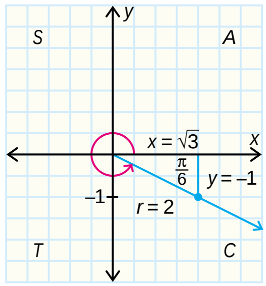
Solve $2 \cos \theta = \sqrt{3}$, $0 \leq \theta \leq 2\pi$.

Solution

$$2 \cos \theta = \sqrt{3} \quad \text{Rearrange the equation by isolating } \cos \theta.$$
$$\cos \theta = \frac{\sqrt{3}}{2}$$

Since $\cos \theta = \frac{x}{r}$, then $x = \sqrt{3}$, and $r = 2$. These are sides in the $\frac{\pi}{3} - \frac{\pi}{6}$ special triangle. The cosine function is positive in quadrants I and IV. There are two possible solutions.





$$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

The solutions are $\theta = \frac{\pi}{6}$ and $\theta = \frac{11\pi}{6}$.

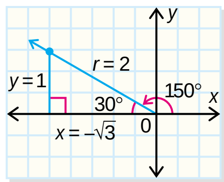
Verifying a Relationship

The special triangles can also be used to show that a relationship is true, if the expression involves 30°, 45°, or 60° angles or related angles.

Example 6

Show that $\frac{\cos 150°}{\sin 150°} = \frac{1}{\tan 150°}$.

Solution



Sketch the angle in standard position. Then determine the values of x, y, and r using the 30° special triangle and the fact that the angle lies in quadrant II.

From the diagram,

$$\cos 150° = \frac{-\sqrt{3}}{2} \qquad \sin 150° = \frac{1}{2} \qquad \tan 150° = \frac{-1}{\sqrt{3}}$$

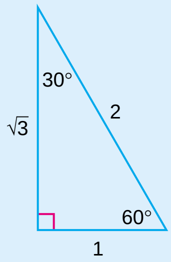
Substitute into the left and right sides of the original expression to verify.

L.S.	R.S.
$\frac{\cos 150°}{\sin 150°}$	$\frac{1}{\tan 150°}$
$= \frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}}$	$= \frac{1}{\frac{1}{-\sqrt{3}}}$
$= \left(\frac{-\sqrt{3}}{2}\right)\left(\frac{2}{1}\right)$	$= 1 \times \frac{-\sqrt{3}}{1}$
$= -\sqrt{3}$	$= -\sqrt{3}$

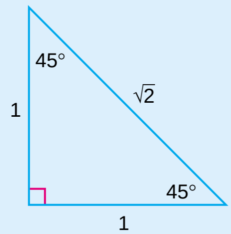
L.S. = R.S., therefore $\frac{\cos 150°}{\sin 150°} = \frac{1}{\tan 150°}$.

Key Ideas

- Use the triangle shown to find the exact value of the trigonometric functions for any angle in standard position where 30° $\left(\frac{\pi}{6}\right)$ or 60° $\left(\frac{\pi}{3}\right)$ is the related angle.



- Use the triangle shown to find the value of the trigonometric functions for any angle in standard position where 45° $\left(\frac{\pi}{4}\right)$ is the related angle.



- To find the exact value of a trigonometric function for any angle that has a related angle of 30°, 45°, or 60°, follow these steps.
  1. Sketch the angle in standard position.
  2. Create a right triangle by drawing a vertical line from the terminal arm of the angle to the x-axis.
  3. Determine the measure of the related angle.
  4. Assign values for x, y, and r using the appropriate special triangle and the quadrant location of the angle.
  5. Determine the exact value using the appropriate trigonometric ratio.
- Sometimes, if a trigonometric equation involves a ratio from one of the special triangles, the solutions can be determined using $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ as related angles.

Practise, Apply, Solve 6.3

A

1. (a) Draw a triangle with angles of 30°, 60°, and 90°. Label the sides using the lengths $\sqrt{3}$, 2, and 1.

(b) Explain your reasoning for positioning each side length.

2. (a) Draw a single triangle with angles of 45°, 45°, and 90°. Label the sides using the lengths $\sqrt{2}$, 1, and 1.

(b) Explain your reasoning for positioning each side.

3. Determine the exact value.

(a) $\cos(-30°)$ (b) $\tan(-60°)$ (c) $\sin(-45°)$

4. Determine the exact value.

(a) $\tan(-150°)$ (b) $\sin(-240°)$ (c) $\cos(-315°)$

5. Determine the exact value.

(a) $\sin \frac{3\pi}{4}$ (b) $\tan \frac{4\pi}{3}$ (c) $\cos \frac{11\pi}{6}$ (d) $\tan \frac{-7\pi}{6}$

B

6. Determine the exact value.

(a) $\sin 495°$ (b) $\cos 570°$ (c) $\tan(-690°)$ (d) $\sin 675°$

7. Determine the exact value.

(a) $\cos \frac{13\pi}{4}$ (b) $\cos \frac{-19\pi}{6}$ (c) $\tan \frac{7\pi}{3}$ (d) $\cos \frac{23\pi}{6}$

8. Use special triangles to show each equation is true.

(a) $\sin 45° = \sin 135°$ (b) $\sin 225° = \sin(-45°)$

(c) $\cos(-45°) = \cos 45°$ (d) $\tan 135° = \tan(-45°)$

9. Use special triangles to show each equation is true.

(a) $\sin 60° = \sin(-240°)$ (b) $\cos(-30°) = \cos 30°$

(c) $\tan 120° = \tan(-60°)$ (d) $\sin 150° = \sin(-330°)$

10. Show that $(\sin\theta)^2 + (\cos\theta)^2 = 1$ for each angle.

(a) $\theta = 45°$ (b) $\theta = \frac{\pi}{3}$ (c) $\theta = 60°$ (d) $\theta = \frac{7\pi}{4}$

11. Show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ for each angle.

(a) $\theta = 120°$ (b) $\theta = -150°$ (c) $\theta = \frac{11\pi}{3}$ (d) $\theta = \frac{13\pi}{4}$

12. Use special triangles to determine the roots of each equation, $0° \le \theta \le 360°$.

(a) $\tan \theta = -1$

(b) $\sin \theta = \frac{1}{\sqrt{2}}$

(c) $\cos \theta = \frac{-\sqrt{3}}{2}$

(d) $\sin \theta = \frac{\sqrt{3}}{2}$

(e) $\cos \theta = \frac{1}{\sqrt{2}}$

(f) $\tan \theta = -\sqrt{3}$

(g) $\sin \theta = \frac{1}{2}$

(h) $\tan \theta = \frac{-1}{\sqrt{3}}$

(i) $\sin 2x = \frac{1}{\sqrt{2}}$

(j) $\cos 2x = \frac{1}{2}$

(k) $5 \cos x - \sqrt{3} = 3 \cos x$

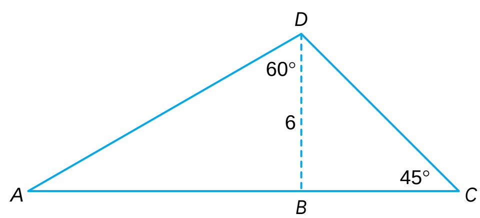
(l) $\sin \frac{x}{2} = \frac{\sqrt{3}}{2}$

13. Graph $y = \sin \theta$, $-360° \le \theta \le 360°$, using 45° increments. Determine the exact coordinates when $\theta = 45° + 90°n$, $n \in \mathrm{I}$. Mark the coordinates on the graph. Explain why $y = \sin \theta$ is a function.

14. Graph $y = \cos \theta$, $-2\pi \le \theta \le 2\pi$, in increments of $\frac{\pi}{6}$. Determine the exact coordinates for $[\theta, f(\theta)]$ if the related angle is $\frac{\pi}{6}$. Mark the coordinates on the graph. Explain why $y = \cos \theta$ is a function.

15. **Check Your Understanding**

(a) Determine the exact measure of each unknown side length in the diagram.



(b) Find the exact value of the sine, cosine, and tangent of $\angle A$ and $\angle C$.