5.8 Solving Linear Trigonometric Equations

The equation $\cos \theta = 0.5$ for the interval $0 \le \theta \le 2\pi$ is a **linear trigonometric equation**. The solutions to this type of equation are all the values in the given domain that make the equation true. In this case, the equation has two solutions, $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. Solutions can be expressed in either degree measure or radian measure. Generally, express solutions using the same notation that is used in the given domain of the equation.

Part 1: Investigating the Number of Solutions

The monthly sales of lawn equipment can be modelled by the function $S(t) = 32.4 \sin\left(\frac{\pi}{6}\right)t + 53.5$, where S is the monthly sales in thousands of units and t is the time in months, $t = 1$ corresponds to January.

Think, Do, Discuss

1. Use a graphing calculator and set the mode to radian.
2. Enter the sales function into Y1 of the equation editor. Ensure that you enter the equation as shown above.
3. Adjust the window. What must be the minimum and maximum settings for the x- and y-axes so that you will see the first year of the function in the graph?
4. Set the window to display the graph for one year and graph the function by using **ZoomFit**. (Press [ZOOM] [0].)
5. What is the period of this function?
6. In which month are monthly sales at a maximum? at a minimum?
7. In which month will 70 000 units be sold? Write the trigonometric equation that corresponds to this situation. Enter 70 into Y2 of the equation editor and graph.
8. How many points of intersection are there? What are they and what do they mean?
9. How many times will the company sell 70 000 units over the next three years? In which months will this occur?
10. Check your predictions by adjusting the window settings for a period of three years and then graph.

11. According to this model, how many times will the company sell 70 000 units over the next ten years?

12. How many solutions are possible for the equation $70 = 32.4 \sin\left(\frac{\pi}{6}\right)t + 53.5$? Explain.

Part 2: Solving Trigonometric Equations

The domain plays an important role in the solution of trigonometric equations. The domain limits the number of possible solutions. Without a domain, a trigonometric equation would have an infinite number of solutions due to the periodic nature of the trigonometric functions.

You can solve a trigonometric equation in several ways.

Estimating the Solutions from a Graph

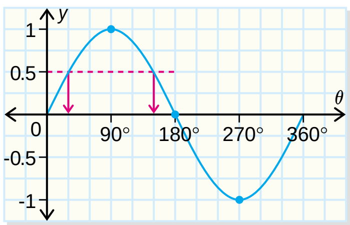
Sometimes you need only an estimate of the solution. To estimate a solution, use the information in a graph.

Example 1

Solve $\sin \theta = 0.5$ for $0° \leq \theta \leq 360°$.

Solution

Sketch the graph of $y = \sin \theta$ for $0° \leq \theta \leq 360°$.



Draw a horizontal line through 0.5 on the y-axis to determine the angles that correspond with this value. From the graph, you can estimate that the solutions are $\theta = 30°$ and $\theta = 150°$.

Using a Scientific Calculator or a Graphing Calculator

You can determine all of the solutions in the given domain by using related angles and the period of the related function. Use a scientific calculator or a graphing calculator to calculate angles.

Example 2

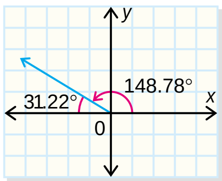
Solve $\cos\theta = -0.8552$ for $0° \le \theta \le 360°$.

Solution

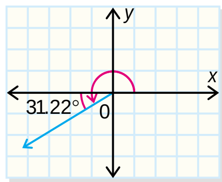
The domain is in degree measure. Ensure that the calculator is in degree mode. The cosine function is negative in quadrants II and III. For the given domain, there are only two solutions. Use a scientific or a graphing calculator to determine $\cos^{-1}(-0.8552)$.

$$\theta = \cos^{-1}(-0.8552)$$
$$\theta \doteq 148.78°$$

This angle is in quadrant II. The related angle is $180° - 148.78° = 31.22°$.



The angle in quadrant III is $180° + 31.22° = 211.22°$.



The two solutions of the equation are 148.78° and 211.22°.

Example 3

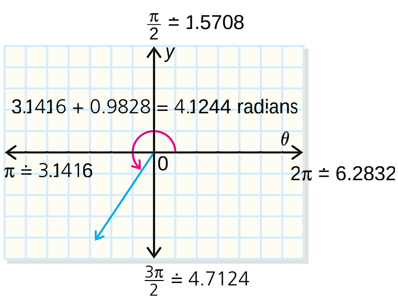
Solve $\tan\theta = 1.5$ for $0 \le \theta \le 2\pi$.

Solution

The domain is given in radian measure. Ensure that the calculator is in radian mode. The tangent function is positive in quadrants I and III. For the given domain, there are only two solutions. Use a scientific or a graphing calculator to determine $\tan^{-1}(1.5)$.

$$\theta = \tan^{-1}(1.5)$$
$$\theta \doteq 0.9828$$

This angle is in quadrant I.



The angle in quadrant III is $\pi + 0.9828 \doteq 3.1416 + 0.9828$ or 4.1244 radians. The two solutions for this equation are 0.9828 radians and 4.1244 radians.

A few more steps are required when the period of a trigonometric equation is not 360° or 2π.

Example 4

Solve $3 \sin (2x) + 2 = 1$ for $0° \le x \le 360°$.

Solution

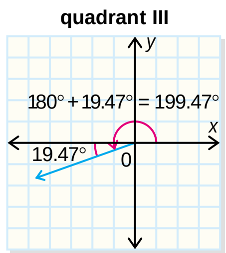
$$
\begin{aligned}
3 \sin (2x) + 2 &= 1 \quad \text{Rearrange the equation by solving for } \sin (2x).\\
3 \sin (2x) &= 1 - 2\\
3 \sin (2x) &= -1\\
\sin (2x) &= -\frac{1}{3}
\end{aligned}
$$

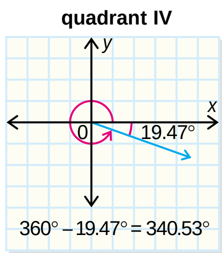
Use a scientific or a graphing calculator to determine $\sin^{-1}\left(-\frac{1}{3}\right)$.

$$\sin^{-1}\left(-\frac{1}{3}\right) \doteq -19.47°$$

This angle is not in the given domain.

The sine function is negative in quadrants III and IV. The related acute angle is 19.47°.





$$
\begin{aligned}
2x &= 199.47°\\
x &= 99.735°
\end{aligned}
\qquad
\begin{aligned}
2x &= 340.53°\\
x &= 170.265°
\end{aligned}
$$

But the function $y = \sin (2x)$ has a period of 180°. Adding 180° to each solution yields the solutions for the second cycle, 180° to 360°. This equation has four solutions.

$x_1 = 99.735°$, $x_2 = 170.265°$, $x_3 = 99.735° + 180°$ or $279.735°$, and $x_4 = 170.265° + 180°$ or $350.265°$

Finding the Point of Intersection Between Two Functions

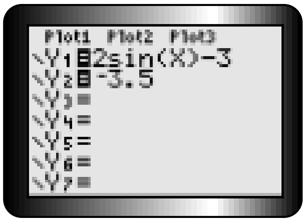
Example 5

Solve $2 \sin \theta - 3 = -3.5$ for $0° \le \theta \le 360°$.

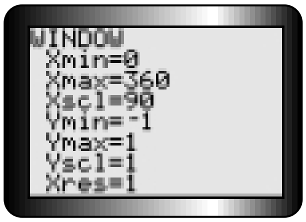
Solution

Find the solutions to this equation by graphing two functions, $y = 2 \sin \theta - 3$ and $y = -3.5$, over the given domain on the same axes. The points of intersection between the two curves are the solutions to the original equation, $2 \sin \theta - 3 = -3.5$.
Put the calculator in degree mode.

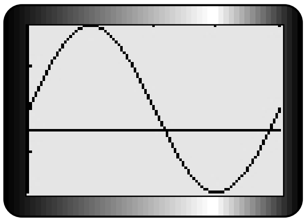
1. Enter the functions into the equation editor.



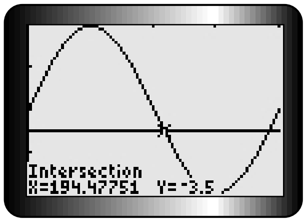
2. Set the window for the given domain.



3. Graph by using ZoomFit. Press [ZOOM] [0].



4. Determine all the points of intersection. Press [2nd] [TRACE] [5] and respond to the questions.



Determine the other point of intersection. The equation $2 \sin(\theta) - 3 = -3.5$ has solutions $\theta = 194.48°$ and $\theta = 345.52°$ for $0° \le \theta \le 360°$.

Using the Zeros of the Corresponding Function

Example 6

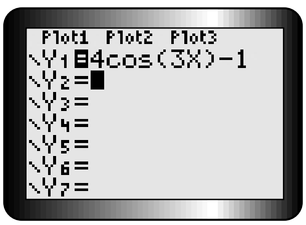
Solve $4 \cos(3x) = 1$ for $0 \le x \le 2\pi$.

Solution

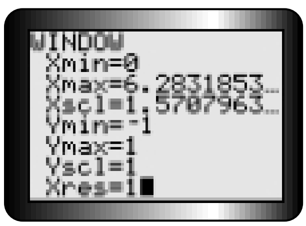
Find the solutions to this equation by rearranging the original equation so that $4 \cos(3x) - 1 = 0$. Graph the corresponding function $y = 4 \cos(3x) - 1$ over the domain $0 \le x \le 2\pi$. The zeros or x-intercepts of $y = 4 \cos(3x) - 1$ correspond to the solutions of $y = 4 \cos(3x) - 1$.

Put the calculator in radian mode before graphing.

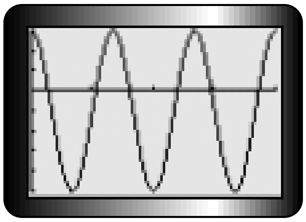
1. Enter the function into the equation editor.



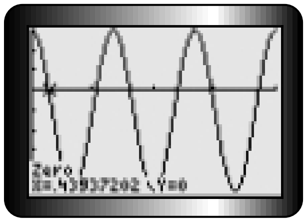
2. Set the window for the given domain. Xmin=0, Xmax=2π, Xscl=$\frac{\pi}{2}$.



3. Graph by using ZoomFit. Press [ZOOM] [0].



4. Determine all the zeros. Press [2nd] [TRACE] [2] and respond to the questions.



Determine the other five zeros of the function $y = 4\cos(3x) - 1$. The equation $4\cos(3x) = 1$ has solutions $x = 0.4394, 1.655, 2.5334, 3.7494, 4.6282$, and 5.8438, where $0 \leq x \leq 2\pi$.

Consolidate Your Understanding

1. Give an example of a linear trigonometric equation and its solutions for a specific domain.
2. How many solutions are possible for a linear trigonometric equation if the domain is not specified?
3. Which technique described in this section is best for solving the majority of linear trigonometric equations?
4. Does a scientific or a graphing calculator always yield *all* the solutions to a linear trigonometric equation over a given domain? Explain.
5. How can you solve *any* linear trigonometric equation?

Focus 5.8

Key Ideas

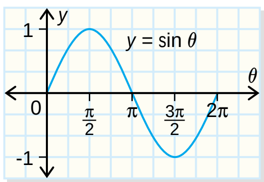
- Any equation that has $\sin x$, $\cos x$, or $\tan x$ and whose highest power is 1 is called a **linear trigonometric equation**. For example, $2\sin x = 1$
- The solutions to a trigonometric equation are all of the values of the independent variable that make the equation a true statement and that are part of a given domain. For example, $2\sin x = 1$ has solutions $x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$, where $0 \leq x \leq 2\pi$ is the domain.
- A trigonometric equation can have an infinite number of solutions due to the periodic nature of trigonometric functions. The given domain limits the number of solutions. For example, the domain $0 \leq x \leq 2\pi$ indicates that the solutions to an equation must lie between these values.
- An approximate solution for any linear trigonometric equation can be interpolated or extrapolated from the graph of the equation.

- Use the inverse trigonometric functions on a scientific or a graphing calculator to obtain reasonable estimates of a solution. However, the solution provided by the calculator may or may not lie in the given domain. Use the related acute angle and the period of the corresponding function to determine all of the solutions in the given domain.
- Estimate the solution for any linear trigonometric equation by using graphing technology.
  i. You can first write the equation as two separate functions, then determine the points of intersection between the two functions over the given domain.
  OR
  ii. First rearrange the trigonometric equation so that zero is on one side, then determine the zeros or *x*-intercepts of the function over the given domain

Practise, Apply, Solve 5.8

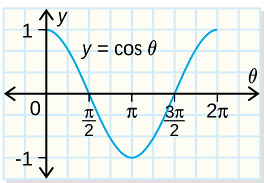
A

1. Use the graph, where the domain is $0 \leq \theta \leq 2\pi$, to determine the value(s) of θ.



(a) $\sin \theta = 1$ (b) $\sin \theta = -1$ (c) $\sin \theta = 0.5$ (d) $\sin \theta = -0.5$

2. Use the graph, where the domain is $0 \leq \theta \leq 2\pi$, to determine the value(s) of θ.



(a) $\cos \theta = 1$ (b) $\cos \theta = -1$ (c) $\cos \theta = 0.5$ (d) $\cos \theta = -0.5$

3. For $\sin x = \frac{\sqrt{3}}{2}$, $0 \leq x \leq 2\pi$.

(a) How many solutions are possible?
(b) In which quadrants would you find the solutions?
(c) Determine the related acute angle for this equation.
(d) Determine all the solutions to the equation.

4. For $\cos x = -0.8667$, $0° \le x \le 360°$.

(a) How many solutions are possible?

(b) In which quadrants would you find the solutions?

(c) Determine the related angle for this equation to the nearest degree.

(d) Determine all the solutions to the equation to the nearest degree.

5. For $\tan \theta = 2.7553$, $0 \le \theta \le 2\pi$.

(a) How many solutions are possible?

(b) In which quadrants would you find the solutions?

(c) Determine the related angle for this equation to the nearest hundredth.

(d) Determine all the solutions to the equation to the nearest hundredth.

B

6. Using a calculator, determine the solutions for each equation. The domain is $0° \le x \le 360°$. Round your answers to the nearest tenth of a degree.

(a) $\sin x = 0.65$ (b) $\cos x = 0.8$ (c) $\tan x = 1.5$

(d) $\sin x = 0.15$ (e) $\tan x = 0.75$ (f) $\cos x = -0.655$

7. Using a calculator, determine the solutions for each equation. The domain is $0 \le \theta \le 2\pi$. Round your answers to the nearest hundredth of a radian.

(a) $\sin \theta = 0.4255$ (b) $\cos \theta = 0\ .1576$ (c) $\tan \theta = 2.35$

(d) $\sin \theta = -0.5005$ (e) $\tan \theta = -0.6341$ (f) $\cos \theta = -0.7583$

8. Using a calculator, determine the roots for each equation. The domain is $0° \le \theta \le 360°$.

(a) $\tan \theta = 1$ (b) $\sin \theta = \frac{1}{\sqrt{2}}$ (c) $\cos \theta = \frac{\sqrt{3}}{2}$

(d) $\sin \theta = -\frac{\sqrt{3}}{2}$ (e) $\cos \theta = -\frac{1}{\sqrt{2}}$ (f) $\tan \theta = \sqrt{3}$

9. Using a calculator, determine the roots for each equation. The domain is $0° \le \theta \le 360°$. Express your answers to one decimal place.

(a) $2 \sin \theta = -1$ (b) $3 \cos \theta = -2$ (c) $2 \tan \theta = 3$

(d) $-3 \sin \theta - 1 = 1$ (e) $-5 \cos \theta + 3 = 2$ (f) $8 - \tan \theta = 10$

10. Knowledge and Understanding

(a) How many solutions does the equation $5 \cos x + 3 = 6$ for $0° \le x \le 360°$ have?

(b) Solve the equation in (a).

11. Using a calculator, determine the solutions for each equation to two decimal places. The domain is $0 \le x \le 2\pi$.

(a) $3 \sin x = \sin x + 1$ (b) $5 \cos x - \sqrt{3} = 3 \cos x$

(c) $\cos x - 1 = -\cos x$ (d) $5 \sin x + 1 = 3 \sin x$

12. Using a calculator, determine the solutions for each equation to two decimal places. The domain is $0 \le x \le 2\pi$.

(a) $\sin 2x = \frac{1}{\sqrt{2}}$

(b) $\sin 4x = \frac{1}{2}$

(c) $\sin 3x = -\frac{\sqrt{3}}{2}$

(d) $\cos 4x = -\frac{1}{\sqrt{2}}$

(e) $\cos 2x = -\frac{1}{2}$

(f) $\cos\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{2}$

13. **Communication:** Sketch the graph of $y = \sin 2\theta$, $0 \le \theta \le 2\pi$. On the graph, clearly indicate all the solutions to the trigonometric equation $\sin 2\theta = -\frac{1}{\sqrt{2}}$.

14. **Thinking, Inquiry, Problem Solving:** Search the Internet to find a sunrise and sunset calculator. Use the calculator to determine the number of hours of daylight over a one-year period for a city near your community. It may be helpful to find the numbers of hours of daylight for 15-day intervals. Use the data to create an algebraic model. Then determine on which day(s) of the year this city gets exactly 11 h of daylight.

15. A Ferris wheel has a radius of 7 m. The centre of the wheel is 8 m above the ground. The Ferris wheel rotates at a constant speed of 15°/s. The height above the ground of the only red seat can be modelled by the function $h(t) = 8 + 7\sin(15°t)$.

(a) Determine the height of the red seat at the start of the ride.

(b) What is the maximum height of any seat?

(c) When is the red seat at its maximum height during the first rotation?

(d) What is the minimum height of any seat?

(e) When is the red seat at its minimum height during the first rotation?

(f) How long will it take for the red seat to complete two full rotations?

16. On a merry-go-round, each horse moves up and down five times in one complete revolution. Imagine that each horse rises and falls 25 cm from its centre position. The up-and-down motion of each horse can be modelled by the function $h(t) = 25\cos(5\theta)$, where h is the horse's displacement from its centre and θ is the rotation angle of the merry-go-round. Assume the ride begins when $\theta = 0°$ for a given horse.

(a) Determine the displacement of one horse at the start of the ride.

(b) At what rotation angles will the horse be displaced 15 cm in one revolution?

(c) At what rotation angles will the horse be displaced −20 cm in one revolution?

(d) How long will it take for one complete revolution if the carousel rotates at a speed of 24°/s?

17. Solve each of the following using graphing technology. The domain is $0° \le x \le 360°$. Express your answers to the nearest tenth of a degree.

(a) $4 \sin x - 3 = -4$

(b) $0.5 \cos x = 0.1$

(c) $3 \tan x - 2 = 5$

(d) $2 \sin (2x) = -1$

(e) $-1 \cos (3x) = 1$

(f) $5 \tan \left(\frac{x}{2}\right) = 1$

(g) $2 \sin (x + 90°) = 1$

(h) $\cos (x - 60°) + 2 = 3$

18. Solve each of the following using graphing technology. The domain is $0 \le x \le 2\pi$. Express your answers to the nearest hundredth.

(a) $3 \sin x + 6 = 5$

(b) $2 \cos (0.5x) = 2$

(c) $2 \tan (2x) = 1$

(d) $4 \sin 3x = -2$

(e) $-2 \cos x = 1$

(f) $-\tan (x) = 5$

(g) $\sin (x - \pi) = 1$

(h) $2 \cos (x + 1) = 0.5$

19. **Application:** The horizontal distance, d, in metres, of a baseball's path when it is hit by a bat can be modelled by $d(\theta) = \frac{v^2}{9.8} \sin (2\theta) + 0.5$, where v is the initial speed of the ball in metres per second and θ is the angle at which the ball leaves the bat.

(a) A home run travels a horizontal distance of 142 m. The ball leaves the bat an angle of 43°. Determine the initial speed of the ball.

(b) The ball travels a horizontal distance of 59 m and leaves the bat at a speed of 30 m/s. Determine the angle at which the ball was struck if

i. it was a line drive

ii. it was a fly ball

20. The average monthly temperature of Littletown can be modelled by the function $T(t) = 14.6 \sin 0.5(t - 1) + 9.15$, where T is the temperature in degrees Celsius and $t = 0$ represents January 1, $t = 1$ represents February 1, and so on.

(a) In which month is the average monthly temperature the highest? the lowest?

(b) Use the model to predict when the temperature is 0°C.

(c) When is the temperature 20°C?

21. The daily mean temperature is the average of the highest and lowest temperatures during one day. The table shows the average daily mean temperature for each month, in degrees Celsius, for London, Ontario, over a 50-year period.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Daily Mean Temperature (°C)	−6.7	−6.2	−0.5	6.2	12.6	17.7	20.3	19.3	15.3	9.1	3.3	−3.4

Source: Environment Canada

(a) Create a scatter plot of temperature versus time, where $t = 1$ represents January, $t = 2$ represents February, and so on.

(b) Draw the curve of best fit.

(c) Determine the trigonometric function that models this relationship.

(d) When will the daily mean temperature be 15°C in London, according to the function? Explain any differences between this and the table.

22. The table shows the average number of monthly hours of sunshine for Toronto.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Average Monthly Sunshine (h)	95.5	112.6	150.5	187.7	229.7	254.9	278	244	184.7	145.7	82.3	72.6

Source: Environment Canada

(a) Create a scatter plot of the number of hours of sunshine versus time, where $t = 1$ represents January, $t = 2$ represents February, and so on.

(b) Draw the curve of best fit.

(c) Determine the trigonometric function that models this relation.

(d) When will the number of monthly hours of sunshine be at a maximum according to the function? When will it be a minimum according to the function?

(e) How good a model is the equation? Explain.

23. Check Your Understanding

(a) For $2 \cos \theta + 3 = 4$, how many solutions are possible if the domain is $0 \leq \theta \leq 2\pi$?

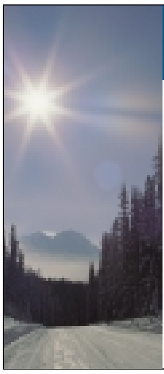
(b) Solve the equation $2 \cos \theta + 3 = 4$ for $0 \leq \theta \leq 2\pi$, without using a calculator or graphing technology.

(c) Verify the solutions you found for the equation in (b) by using graphing technology.

C

24. Solve $\cos \theta - \sin \theta = 0$ for $0 \leq \theta \leq 2\pi$, without using graphing technology.

25. Solve $\sin (2\theta - 20°) = \frac{1}{2}$ for $0° \leq \theta \leq 360°$, without using graphing technology.



The Chapter Problem—How Much Daylight?

In this section, you studied linear trigonometric equations. Apply what you learned to answer this question on the Chapter Problem on page 404.

CP14. The equation $h(t) = 7 \sin \frac{\pi}{6}(t - 2) + 12.5$ is a model for the average number of hours of daylight per month, beginning at the turn of the millennium. Solve for t if $h(t) = 13$ and $0 \leq t \leq 240$.