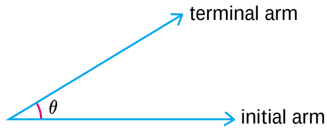
Understanding Angles

A triangle has three angles and no angle can be equal to or greater than 180°. Consider what happens when an angle is not part of a triangle but is in the *x*-*y* plane.

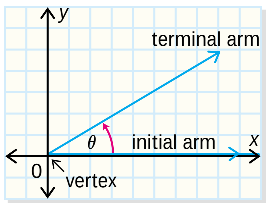
Angles and Their Location in the *x*-*y* Plane

An angle is formed when a ray is rotated about a fixed point called the **vertex**. The ray is called the **initial arm** at the beginning of the angle and the **terminal arm** at the end of the angle. Angles are often labelled with Greek letters, such as θ "theta," α "alpha," and β "beta."



An angle θ is in **standard position** if the vertex of the angle is at the origin and the initial arm lies along the positive *x*-axis. The terminal arm can be anywhere on the arc of rotation.

Standard Form



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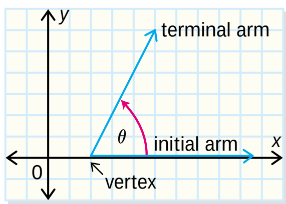

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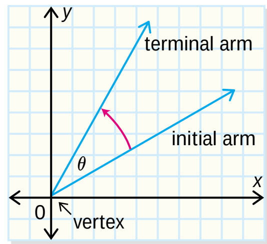
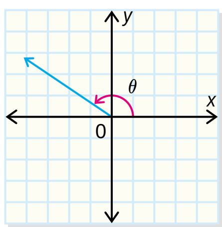


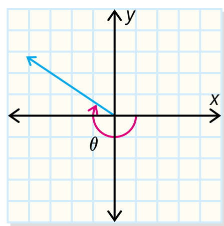
An angle can be positive or negative.

A **positive** angle is formed by a counter-clockwise rotation of the terminal arm.



a positive angle

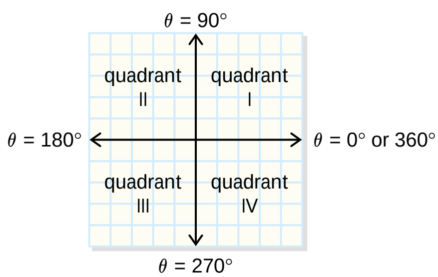
A **negative** angle is formed by a clockwise rotation of the terminal arm.



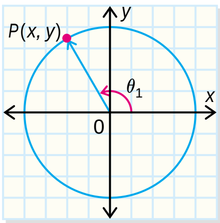
a negative angle

The *x*-*y* plane is divided into four **quadrants** by the *x*- and *y*-axes. If θ is a positive angle, then the terminal arm lies in

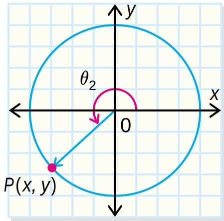
- quadrant I when $0° < \theta < 90°$
- quadrant II when $90° < \theta < 180°$
- quadrant III when $180° < \theta < 270°$
- quadrant IV when $270° < \theta < 360°$



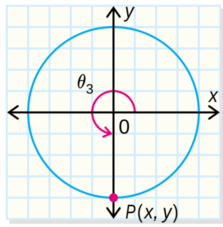
Let $P(x, y)$ be a point on the terminal arm of an angle in standard position. Since P can be anywhere in the x-y plane, the angle can terminate anywhere in the x-y plane.



$90° < \theta_1 < 180°$
θ_1 terminates in quadrant II.

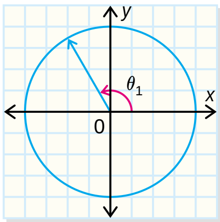


$180° < \theta_2 < 270°$
θ_2 terminates in quadrant III.

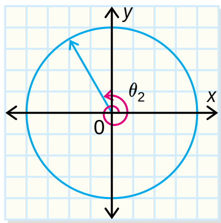


$P(x, y)$ lies in the negative y-axis.
$\theta_3 = 270°$

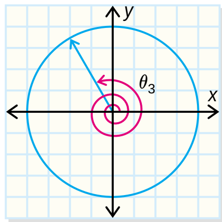
Coterminal angles share the same terminal arm and the same initial arm. As an example, here are four different angles with the same terminal arm and the same initial arm.



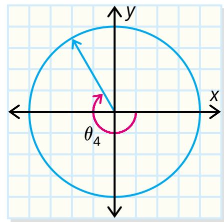
If $\theta_1 = 120°$, then



$\theta_2 = 360° + 120°$
$= 480°$



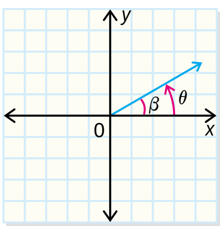
$\theta_3 = 720° + 120°$
$= 840°$

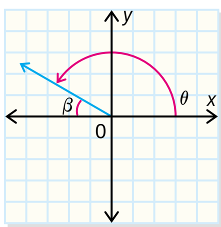


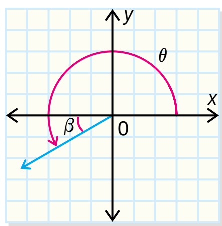
$\theta_4 = -360° + 120°$
$= -240°$

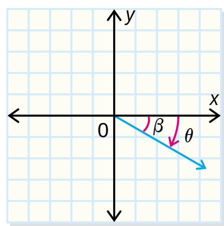
The **principal angle** is the angle between 0° and 360°. The coterminal angles of 480°, 840°, and −240° all share the same principal angle of 120°.

The **related acute angle** is the angle formed by the terminal arm of an angle in standard position and the x-axis. The related acute angle is always positive and lies between 0° and 90°. In this example, β represents the related acute angle for θ.







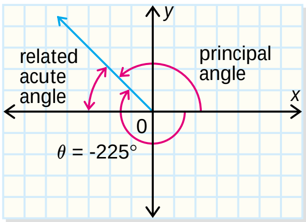


Example 1

Determine the principal angle and the related acute angle for $\theta = -225°$.

Solution

Sketch $\theta = -225°$ terminating in quadrant II. Label the principal angle and the related acute angle.



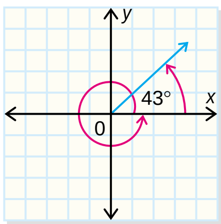
The principal angle is the smallest positive angle that is coterminal to $-225°$. In this case, $360° - 225° = 135°$. The related acute angle lies between the terminal arm and the x-axis. It is positive but less than 90°. In this case, $|-225° - (-180°)| = 45°$. Or, using the principal angle, $180° - 135° = 45°$.

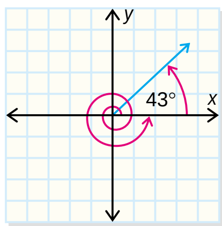
Example 2

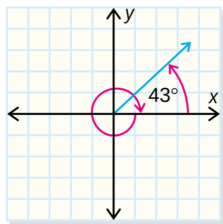
Determine the next two consecutive positive coterminal angles and the first negative coterminal angle for 43°.

Solution

Sketch each situation, showing the principal angle of 43°.







(a) The first positive coterminal angle for 43° is $360° + 43° = 403°$.

(b) The second coterminal angle is $360° + 360° + 43° = 763°$.

(c) The first negative coterminal angle is $-360° + 43° = -317°$.

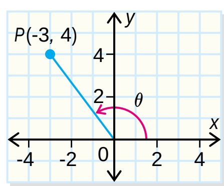
Example 3

Point $P(-3, 4)$ is on the terminal arm of an angle in standard position.

(a) Sketch the principal angle, θ.

(b) Determine the value of the related acute angle to the nearest degree.

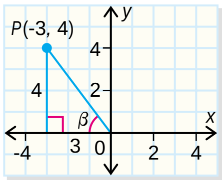
(c) What is the measure of θ to the nearest degree?



Solution

(a) Point $P(-3, 4)$ is in quadrant II, so the principal angle, θ, terminates in quadrant II.

(b) The related acute angle, β, is in the right triangle.



The opposite side and the adjacent side are known so the tangent ratio can be used.

$$\tan \beta = \frac{\text{opposite}}{\text{adjacent}} \qquad \text{Substitute known values.}$$

$$\tan \beta = \frac{4}{3}$$

$$\beta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\beta \doteq 53°$$

(c)

$$\begin{aligned} \theta &= 180° - \beta \\ &= 180° - 53° \\ &= 127° \end{aligned}$$

Focus 5.2

Key Ideas

- Angles can be located anywhere in the *x-y* plane.
- The *x*- and *y*-axes divide the *x-y* plane into four quadrants.
- The vertex of an angle in standard position is at the origin, and the initial arm of the angle is along the positive *x*-axis. The terminal arm of the angle can lie anywhere in the *x-y* plane.
- The initial arm of an angle rotates to its terminal position, either in a positive, counterclockwise direction or a negative, clockwise direction.
- The **principal angle** is the first positive angle less than 360°.
- The terminal arm of an angle defines an infinite number of coterminal angles. These can be positive or negative and are defined in terms of the principal angle. They are multiples of 360°; that is, $360°n$, where $n \in I$.
- The **related acute angle** is the positive angle between the terminal arm and the *x*-axis. It is always less than 90°.
- Any angle in standard position can be expressed in terms of its related acute angle.

Practise, Apply, Solve 5.2

A

1. Sketch each angle in standard position.

(a) 135° (b) 210° (c) 315° (d) −30°
(e) −225° (f) −330° (g) 150° (h) −120°
(i) 105° (j) −163° (k) 321° (l) −280°

2. Determine the related acute angle for each angle in question 1.

3. Sketch each angle in standard position.

(a) 379° (b) 491° (c) −545° (d) −640° (e) 593°

4. Determine whether each pair of angles is coterminal or not.

(a) 23°, 383° (b) 41°, 421° (c) −50°, 310°
(d) 38°, 398° (e) −19°, 390° (f) −41°, 319°
(g) 28°, −232° (h) −105°, −465° (i) −123°, 237°
(j) −190°, 180°

5. Calculate the next two positive coterminal angles.

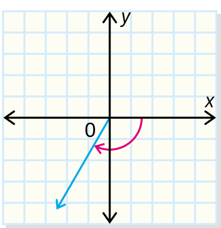
(a) 132° (b) 275° (c) 305° (d) 73° (e) 270°

6. Calculate the next two negative coterminal angles.

(a) −53° (b) −138° (c) −299° (d) −180° (e) −192°

7. Match each angle with its diagram.

(a) 150° (b) −120° (c) 765° (d) −650°
(e) −220° (f) 290° (g) 560° (h) −380°

i.


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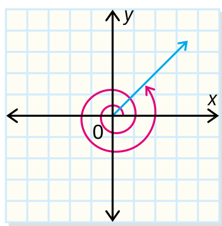

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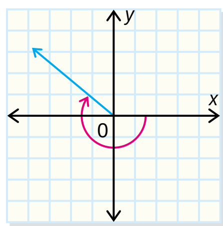

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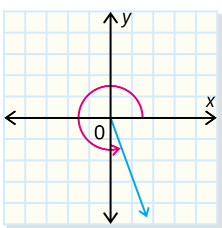

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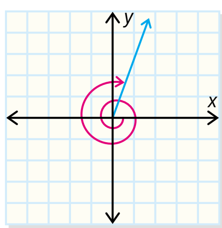

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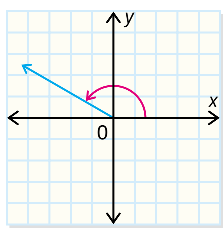
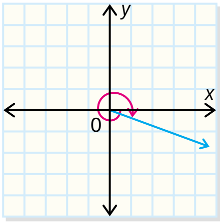
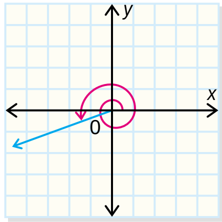


vii. 

viii. 

8. Determine the principal angle.

(a) $-187°$ (b) $410°$ (c) $-67°$ (d) $905°$

(e) $-282°$ (f) $-730°$ (g) $135°$ (h) $1249°$

9. State the principal angle for the given related acute angle and given quadrant.

(a) 24°, quadrant II (b) 35°, quadrant III

(c) 19°, quadrant IV (d) 63°, quadrant I

B

10. State all values of θ, where $n \in I$ as shown.

(a) $\theta = 51° + 360°n,\ 4 \leq n \leq 6$ (b) $\theta = -71° + 360°n,\ -1 \leq n \leq 2$

(c) $\theta = -123° + 360°n,\ -2 \leq n \leq 0$ (d) $\theta = 195° + 360°n,\ 5 \leq n \leq 7$

11. Point $P(-9, 4)$ is on the terminal arm of an angle in standard position.

(a) Sketch the principal angle, θ.

(b) What is the measure of the related acute angle to the nearest degree?

(c) What is the measure of θ to the nearest degree?

12. Point $P(7, -24)$ is on the terminal arm of an angle in standard position.

(a) Sketch the principal angle, θ.

(b) What is the measure of the related acute angle to the nearest degree?

(c) What is the measure of θ to the nearest degree?

13. Point $P(-5, -3)$ is on the terminal arm of an angle, θ, in standard position.

(a) Sketch the principal angle, θ.

(b) What is the measure of the related acute angle to the nearest degree?

(c) What is the measure of θ to the nearest degree?

(d) What is the measure of the first negative coterminal angle?

14. **Check Your Understanding:** Point $P(-5, -9)$ is on the terminal arm of an angle θ in standard position. Explain the role of the right triangle and the related acute angle in determining the principal value of θ.

C

15. Point $P(-5, -8)$ is on the terminal arm of an angle, θ, in standard position. Determine all values of θ for $-540° \leq \theta \leq 270°$.