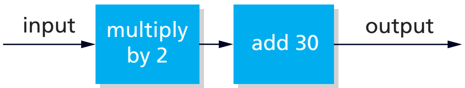
The Inverse Function 3.4

Part 1: Defining the Inverse Function

In grade 10, you used trigonometry to find sides and angles in triangles. For a right triangle, $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$. You saw that on a calculator, [SIN] can be used to find the value of this ratio for a given angle. You also saw that on a calculator, [SIN⁻¹] can be used to find which angle has the value of a given ratio. [SIN⁻¹] is the reverse of [SIN]. In other words, [SIN⁻¹] is the **inverse function** of [SIN]. In this section, you will learn about the concepts, notation, and properties of inverse functions.

Think, Do, Discuss

1. The formula for converting a temperature in degrees Celsius into degrees Fahrenheit is $F = \frac{9}{5}C + 32$. An American visitor to Canada uses this simpler rule to convert from Celsius to Fahrenheit: double the Celsius temperature, then add 30.



Skating on the Rideau Canal in Ottawa

(a) Copy and complete the table using the visitor's rule.

Temperature (°C)	Temperature (°F)
10	50
15	
20	
25	
30	

(b) What is the independent variable? the dependent variable?

(c) Does this rule define a function? Explain.

(d) Let f represent the rule. What ordered pair, (0, ■), belongs to f?

(e) Let x represent the temperature in degrees Celsius. Write the equation for this rule in function notation.

(f) Graph the relation. Use the same scale of −40 to 100 on each axis.

(g) In the table, $f(10) = 50$, which corresponds to a point on the graph of $y = f(x)$. What is the x-coordinate of this point? What is its y-coordinate?

2. A Canadian visited Florida and used this rule to convert the temperature from degrees Fahrenheit into degrees Celsius. To convert 50°F into a temperature in degrees Celsius, the Canadian subtracted 30 and divided the result by 2 to get 10°C.

(a) Copy and complete the table using this rule.

Temperature (°F)	Temperature (°C)
50	10
60	
70	
80	
90	

(b) What is the independent variable? the dependent variable?

(c) This relationship is called the **inverse** of the function in step 1. The first function converts from degrees Celsius into degrees Fahrenheit. The inverse function is the "reverse" of the original function because it converts from degrees Fahrenheit into degrees Celsius. Compare the tables in steps 1(a) and 2(a). Describe how you could use a table for a relation to get a table for its inverse relation.

(d) In mathematics, f is often used to denote the original function, and f^{-1} is used to denote the **inverse** function. Notice that $(10, 50) \in f$ and $(50, 10) \in f^{-1}$. Describe how you can find an ordered pair that belongs to the inverse function if you know an ordered pair that belongs to the original function.

(e) What operations will reverse or "undo" the original rule? Write the rule that the Canadian could use to convert temperature from degrees Fahrenheit into degrees Celsius.

(f) Let x represent the temperature in degrees Fahrenheit. Write the equation for this rule in function notation.

(g) Graph the inverse relation on the same axes you drew in step 1(f).

(h) Draw the line with equation $y = x$ on the same axes. Fold your graph paper along the line $y = x$. What do you notice about the graphs of f and f^{-1}?

(i) In the table, $f^{-1}(50) = 10$, which corresponds to a point on the graph of $y = f^{-1}(x)$. What is the x-coordinate of this point? What is the y-coordinate? What is the corresponding point on the graph of $y = f(x)$?

(j) What are the coordinates of the point of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$? What is the significance of this point?

Part 2: Determining an Algebraic Expression for the Inverse Function

In this section, you will explore different ways of finding the equation of $f^{-1}(x)$. In many applications of functions, it is important to convert easily from input to output and from output to the original input. But solving for $f^{-1}(x)$ may be difficult or even impossible in some cases.

When you get ready for school, you probably follow these steps to put on your shoes.

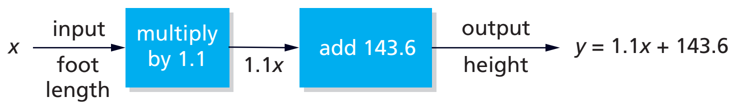
→ put on your socks → pull on your shoes → tie your laces →

When you take off your shoes, you perform the *inverse* steps in the *opposite* sequence.

← take off your socks ← pull off your shoes ← untie your laces ←

Finding an inverse relation is very similar. To find an inverse relation, do the inverse operations in the opposite sequence.

In section 3.1, you found a model for the relationship between footprint length and height by using footprint length as the input and height as the output. An equation for the line of best fit is $y = 1.1x + 143.6$, where x is the footprint length in centimetres and y is the height in centimetres.



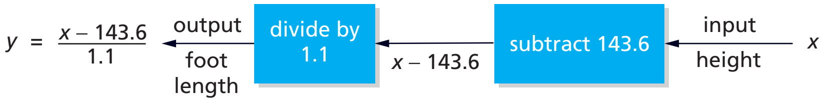
Could you use this relationship to estimate footprint length if you knew a person's height? Suppose the person is 170 cm tall.

$$170 = 1.1x + 143.6$$
$$26.4 = 1.1x$$
$$x = 24$$

A person who is 170 cm tall may have a footprint that is 24 cm long.

So you can use the length of a person's footprint to estimate the person's height. You can also reverse the process. That is, you can use a person's height to estimate the length of the person's footprint.

To get the inverse relation, reverse the operations and their order. So the independent variable (footprint length) becomes the dependent variable and the dependent variable becomes the independent variable.



The equation for the inverse relation is $y = \frac{x - 143.6}{1.1}$.

The equations $y = 1.1x + 143.6$ and $y = \frac{x - 143.6}{1.1}$ both describe the relationship between footprint length and height, but the quantities for the independent and dependent variables have been interchanged.

You can also find the equation of the inverse relation algebraically.

First interchange the independent and dependent variable in the original relation. Then solve for y.

original relation: $y = 1.1x + 143.6$

inverse relation: $x = 1.1y + 143.6$

These equations look almost the same. Solve the inverse relation for y, the output value.

$$x = 1.1y + 143.6 \quad \text{Subtract 143.6 from both sides.}$$
$$x - 143.6 = 1.1y \quad \text{Divide both sides by 1.1.}$$
$$\frac{x - 143.6}{1.1} = y$$

The inverse relation is $f^{-1}(x) = \frac{x - 143.6}{1.1}$ and it gives footprint length as a function of height.

To find the footprint length of someone who is 170 cm tall, substitute $x = 170$ in the inverse relation.

$$f(170) = \frac{170 - 143.6}{1.1}$$
$$= \frac{26.4}{1.1}$$
$$= 24$$

A person who is 170 cm tall may have a footprint that is 24 cm long.

Example 1

(a) For $f(x) = 4x - 8$, determine $f^{-1}(x)$.

(b) Find $f(2)$.

(c) Find $f^{-1}(0)$.

(d) Compare your answers for (b) and (c). Explain what you notice.

Solution

(a) Rewrite $f(x) = 4x - 8$ as $y = 4x - 8$.

To determine $f^{-1}(x)$, interchange x and y and then solve for y.

$$x = 4y - 8$$
$$x + 8 = 4y$$
$$\frac{x+8}{4} = y$$

$\therefore f^{-1}(x) = \frac{x+8}{4}$

(b) $f(2) = 4(2) - 8$
$= 8 - 8$
$= 0$

(c) $f^{-1}(0) = \frac{0+8}{4}$
$= \frac{8}{4}$
$= 2$

(d) In (b), $(2, 0) \in f$. In (c), $(0, 2) \in f^{-1}$. These points seem to correspond, and this makes sense because the x- and y-coordinates have been interchanged.

Example 2

The equation for g is $2x + 3y = 6$. Determine $g(x)$ and $g^{-1}(x)$.

Solution

To determine $g(x)$, first solve for y.

$2x + 3y = 6$	Subtract $2x$ from both sides.
$3y = 6 - 2x$	Divide both sides by 3.
$y = 2 - \frac{2}{3}x$	

$\therefore g(x) = 2 - \frac{2}{3}x$

Then find $g^{-1}(x)$.

g is defined by:	$2x + 3y = 6$	Interchange x and y.
g^{-1} is defined by:	$2y + 3x = 6$	Solve for y.
	$2y = 6 - 3x$	
	$y = 3 - \frac{3}{2}x$	

$\therefore g^{-1}(x) = 3 - \frac{3}{2}x$

Consolidate Your Understanding

1. How can you find the equation of the inverse relation if you know the equation of the relation?
2. How is finding $f^{-1}(x)$ different from finding the inverse of a relation?
3. Describe how you would find $f^{-1}(x)$ if you were given $f(x)$.

Focus 3.4

Key Ideas

- The **inverse** of a relation and a function maps each output of the original relation back onto the corresponding input value. The inverse is the "reverse" of the original relation, or function.
- f^{-1} is the name for the inverse relation.
- $f^{-1}(x)$ represents the expression for calculating the value of f^{-1}.
- If $(a, b) \in f$, then $(b, a) \in f^{-1}$.
- Given a table of values for a function, interchange the independent and dependent variables to get a table for the inverse relation. The domain of f is the range of f^{-1}. The range of f is the domain of f^{-1}.
- The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.
- The graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at points on the line $y = x$.
- $(x, f(x))$ represents any point on the graph of $y = f(x)$.
- $(x, f^{-1}(x))$ represents any point on the graph of $y = f^{-1}(x)$.
- To determine the equation of the inverse in function notation, interchange x and y and solve for y.

Example 3

The table shows all of the ordered pairs belonging to function g.

x	y
1	5
2	7
3	9
4	11
5	13

(a) Determine $g(x)$.

(b) Write the table for the inverse relation.

(c) Evaluate $g(5)$.

(d) Evaluate $g^{-1}(5)$.

(e) What are the coordinates of the point that corresponds to $g^{-1}(5)$ on the graph of g^{-1}?

(f) What are the coordinates of the point on the graph of g that corresponds to $g^{-1}(5)$?

(g) Determine $g^{-1}(x)$.

Solution

(a) The points form a line with slope 2 (each time x increases by 1, y increases by 2). Extrapolate to find that the y-intercept is 3, so the equation of the line is $y = 2x + 3$, where $x \in \{1, 2, 3, 4, 5\}$.

$\therefore g(x) = 2x + 3$, where $x \in \{1, 2, 3, 4, 5\}$

(b) The table for g^{-1} is shown. Note that the x- and y-coordinates are reversed.

x	y
5	1
7	2
9	3
11	5
13	5

(c) From the table for g, $g(5) = 13$.

(d) From the table for g^{-1}, $g^{-1}(5) = 1$.

(e) $g^{-1}(5)$ corresponds to $(5, 1)$ on the graph of g^{-1}.

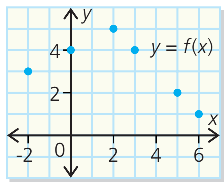
(f) $g^{-1}(5)$ corresponds to $(1, 5)$ on the graph of g.

(g) For g, multiply x by 2 and then add 3. To reverse these operations, subtract 3 and then divide by 2.

$\therefore g^{-1}(x) = \frac{x-3}{2}$, where $x \in \{5, 7, 9, 11, 13\}$

Example 4

The graph of $y = f(x)$ is shown.



i. State the domain and range of f.

ii. Draw an arrow diagram for f^{-1}.

iii. Evaluate.
(a) $f(2)$ **(b)** $f(4)$ **(c)** $f^{-1}(1)$ **(d)** $f^{-1}(4)$

iv. Graph $y = f^{-1}(x)$.

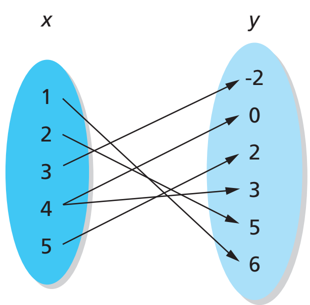
v. Is f^{-1} a function? Explain.

vi. State the domain and range of f^{-1}.

Solution

i. The domain is $\{-2, 0, 2, 3, 5, 6\}$ and the range is $\{1, 2, 3, 4, 5\}$.

ii.



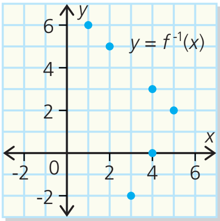
iii. **(a)** $f(2) = 5$, since $(2, 5)$ is on the graph

(b) $f(4)$ is undefined, because there is no point on the graph with x-coordinate 4. The value 4 is not in the domain of f.

(c) $f^{-1}(1) = 6$, because $f(6) = 1$

(d) There are two possible values of $f^{-1}(4)$ because $f(3) = 4$ and $f(0) = 4$.
$f^{-1}(4) = 3$ or $f^{-1}(4) = 0$

iv. Switch x and y in each ordered pair of f and plot these new points.



v. f is a function because each input value has a unique output value. However, the points (0, 4) and (3, 4) belong to f, so (4, 0) and (4, 3) belong to f^{-1}. f^{-1} is not a function, because one input value, 4, has two output values, 0 and 3.

vi. The domain of f^{-1} is {1, 2, 3, 4, 5} and the range is {−2, 0, 2, 3, 5, 6}.

Example 5

A relation is $h(x) = -4x + 6$, where $\{x \mid -2 \leq x \leq 3, x \in \mathbf{R}\}$.

(a) Sketch the graph of $y = h(x)$.

(b) Sketch the graph of $y = h^{-1}(x)$.

(c) State the domain and range of h.

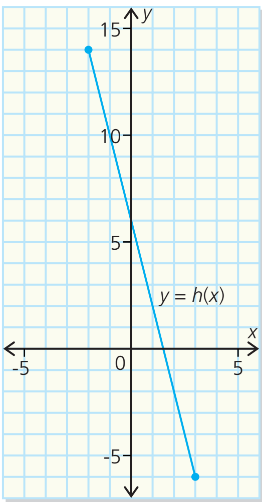
(d) State the domain and range of h^{-1}.

(e) Are h and h^{-1} functions? Explain.

Solution

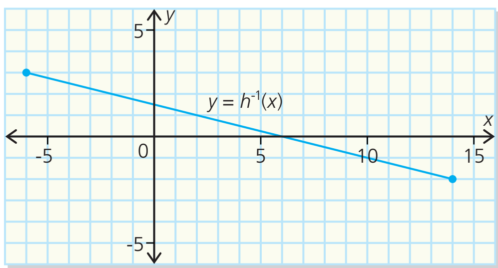
(a)

x	y
−2	14
−1	10
0	6
1	2
2	−2
3	−6



(b) Interchange x and y and plot the new points.

x	y
14	−2
10	−1
6	0
2	1
−2	2
−6	3



(c) The domain of h is $\{x \mid -2 \leq x \leq 3, x \in \mathbf{R}\}$.
The range of h is $\{y \mid -6 \leq y \leq 14, y \in \mathbf{R}\}$.

(d) The domain of h^{-1} is $\{x \mid -6 \leq x \leq 14, x \in \mathbf{R}\}$.
The range of h^{-1} is $\{y \mid -2 \leq y \leq 3, y \in \mathbf{R}\}$.

(e) Both h and h^{-1} are functions. Both pass the vertical line test. For each relation, each value in the domain corresponds with one unique value in the range.

Practise, Apply, Solve 3.4

A

1. For each set of ordered pairs,

i. graph the relationship and its inverse

ii. is the relationship a function? Is the inverse a function? Explain.

(a) $\{(0, 1), (1, 3), (2, 5), (3, 7)\}$

(b) $\{(0, 3), (1, 3), (2, 3), (3, 3)\}$

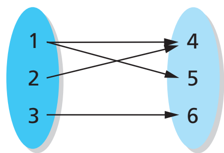
(c) $\{(1, 1), (1, 2), (1, 3), (1, 4)\}$

2. For each of the following,

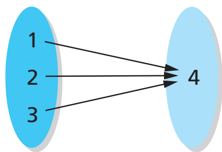
i. draw an arrow diagram for the inverse relationship

ii. state whether or not each inverse defines a function, and justify your answer

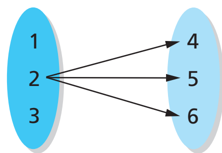
(a)



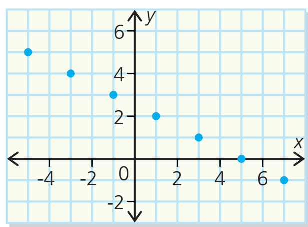
(b)



(c)



3. The graph of the function f is shown.



(a) Create a table of first differences for f.

(b) Create a table of first differences for f^{-1}.

(c) Graph f^{-1}.

(d) Determine the slope of the line that passes through the points belonging to f.

(e) Determine the slope of the line that passes through the points belonging to f^{-1}.

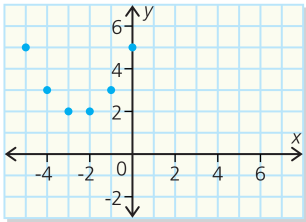
(f) Determine $f(x)$.

(g) Determine $f^{-1}(x)$.

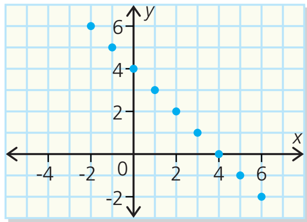
(h) Compare your answers to (f) and (g). Explain.

4. Graph the inverse of each relation.

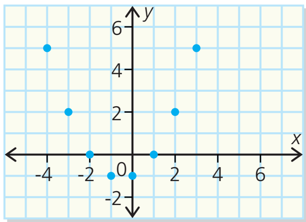
(a)



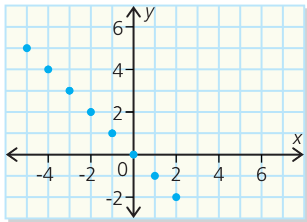
(b)



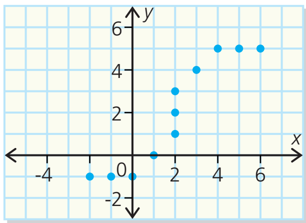
(c)



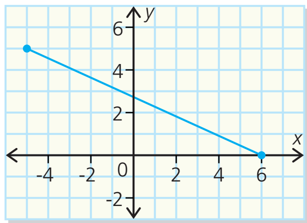
(d)



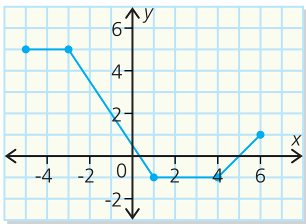
(e)



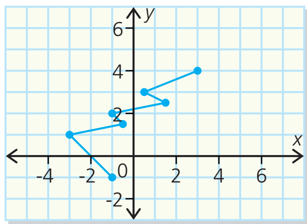
(f)



(g)



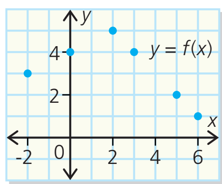
(h)



5. For each part in question 4, identify the points that are common to both the relation and its inverse. Explain.

B

6. The graph of a relation, $y = f(x)$, is shown.



i. Graph $y = f^{-1}(x)$.

ii. State the domain and range of f^{-1}.

iii. Evaluate.

(a) $f(3)$ **(b)** $f^{-1}(3)$ **(c)** $f(2)$

(d) $f^{-1}(2)$ **(e)** $f^{-1}(5)$ **(f)** $f^{-1}(4)$

iv. Is f^{-1} a function? Explain.

v. What point on the graph of f^{-1} corresponds to $f^{-1}(1) = 6$? What coordinate does the value 1 represent at that point? What coordinate does $f^{-1}(1)$ represent?

7. For $g(t) = 3t - 2$, determine each of the following.

(a) $g(0)$
(b) $g^{-1}(-2)$
(c) $g(5)$
(d) $g^{-1}(13)$
(e) $g^{-1}(t)$
(f) $g^{-1}(0)$
(g) $g^{-1}(5)$
(h) $g^{-1}(a)$
(i) $g^{-1}(x - 2)$
(j) $g^{-1}(3a - 2)$
(k) $3g^{-1}(t)$
(l) $3g^{-1}(t) - 2$
(m) $g^{-1}(8) - g^{-1}(7)$
(n) $g^{-1}(14) - g^{-1}(13)$
(o) $g^{-1}(a + 1) - g^{-1}(a)$
(p) $\dfrac{g(13) - g(7)}{13 - 7}$
(q) $\dfrac{g^{-1}(13) - g^{-1}(7)}{13 - 7}$

8. For $f(x) = \frac{2}{3}(x - 5)$, determine each of the following.

(a) $f(-4)$
(b) $f^{-1}(-2)$
(c) $f^{-1}(6)$
(d) $f^{-1}(x)$
(e) $f(x)$
(f) $\frac{2}{3}(f^{-1}(-6) - 5)$
(g) $f^{-1}\left(\frac{8}{3}\right)$
(h) $f^{-1}(a)$
(i) $f^{-1}(a + 1)$
(j) $f^{-1}(a + 1) - f^{-1}(a)$
(k) $\dfrac{2(f^{-1}(t) - 5)}{3}$
(l) $\dfrac{f^{-1}(12) - f^{-1}(4)}{12 - 4}$
(m) $\dfrac{f(17) - f(8)}{17 - 8}$
(n) $\dfrac{f^{-1}(a) - f^{-1}(b)}{a - b}$

9. **Knowledge and Understanding:** The relation f is defined by $2x + 3y = 7$. Graph f and f^{-1}.

(a) Determine $f(x)$.
(b) Determine $f^{-1}(x)$.
(c) Solve $f(x) = 5$.
(d) $f(x) = 5$ corresponds to what point on the graph of f?
(e) $f(x) = 5$ corresponds to what point on the graph of f^{-1}?
(f) Show your answers to (d) and (e) on your graphs.
(g) Solve $f^{-1}(x) = 5$.
(h) $f^{-1}(x) = 5$ corresponds to what point on the graph of f?
(i) $f^{-1}(x) = 5$ corresponds to what point on the graph of f^{-1}?
(j) Show your answers to parts (h) and (i) on your graphs.
(k) Solve $f(x) = f^{-1}(x)$.
(l) $f(x) = f^{-1}(x)$ corresponds to what point on the graph of f?
(m) $f(x) = f^{-1}(x)$ corresponds to what point on the graph of f^{-1}?
(n) Show your answers to (l) and (m) on your graphs.

10. An American visitor to Canada uses this rule to convert from centimetres into inches, "multiply by 4 and then divide by 10." Let the function f be the method for converting centimetres to inches, according to this rule.

(a) Write f^{-1} as a rule.
(b) Describe a situation where the rule for f^{-1} might be useful.

(c) Determine $f(x)$.

(d) Determine $f^{-1}(x)$.

(e) One day, 15 cm of snow fell. Use function notation to represent this amount in inches.

(f) Kelly is about 5 feet 6 inches tall. Use function notation to represent her height in centimetres.

11. For $g(x) = -x + 5$,

(a) find $g(7)$ (b) find $g^{-1}(7)$ (c) find $g^{-1}(x)$

(d) graph g and g^{-1} on the same axes (e) explain your results

12. For $f(x) = 3 - 2x$,

i. graph $f(x)$ and its inverse on the same axes

ii. solve

(a) $f(x) = 0$ (b) $f^{-1}(x) = 0$ (c) $f(x) = x$

(d) $f^{-1}(x) = x$ (e) $f^{-1}(x) = f(x)$

iii. For each part in ii, explain what you are asked to do in terms of the graph of f and f^{-1}.

13. **Communication:** An electronics store pays its employees by commission. The relation $p(s) = 100 + 0.05s$ is used to find an employee's weekly pay, p, in dollars, where s represents the employee's weekly sales in dollars.

(a) Describe the function as a rule.

(b) Determine $p^{-1}(s)$.

(c) Describe the inverse function as a rule.

(d) Describe a situation where the employee might use the inverse function.

(e) State a reasonable domain and range for p^{-1}.

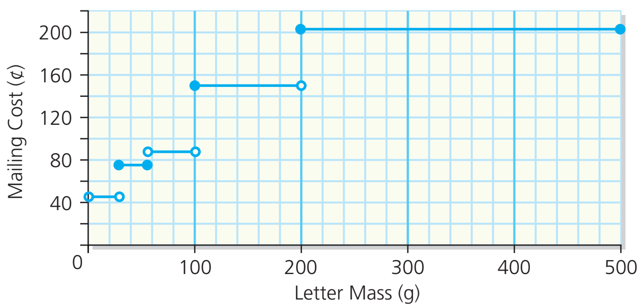
14. The graph shows the cost of mailing a first-class letter in Canada in 2001.

(a) Graph the inverse relation.

(b) Is the inverse relation a function? Explain.

(c) What is the mass of a letter if there is a 94¢ stamp on the letter?

Letter Mass	Mailing Cost
less than 30 g	47¢
30 g to 50 g	75¢
over 50 g but less than 100 g	94¢
over 100 g but less than 200 g	$1.55
200 g to 500 g	$2.05



15. Ali did his homework at school with a graphing calculator. He determined that the equation of the line of best fit for some data was $y = 2.63x - 1.29$. Once he got home, he realized he had mixed up the independent and dependent variables. Write the correct equation for the relation in the form $y = mx + b$.

16. Tiffany is paid \$8.05/h, plus 5% of her sales over \$1000, for a 40-h work week. For example, suppose Tiffany sold \$1800 worth of merchandise. Then she would earn \$8.05(40) + 0.05(\$800) = \$362.

(a) Graph the relation between total pay for a 40-h work week and her sales for the work week.

(b) Write this relation in function notation.

(c) Graph the inverse relation.

(d) Write the inverse relation in function notation.

(e) Write an expression using function notation that represents her sales if she earned \$420 one work week. Then evaluate.

17. **Thinking, Inquiry, Problem Solving:** The manager of the meat department at a grocery store noted that sales of ground beef depended on the price. The table records a range of prices and the corresponding sales.

Price per Kilogram ($)	4.39	4.07	4.65	4.59	3.94
Mass Sold (kg)	1000	2500	700	800	2700

(a) Draw a scatter plot for the relation on your graphing calculator.

(b) Determine an equation of the line of best fit for this relation and write the relation in function notation.

(c) Create a scatter plot for the inverse relation.

(d) Determine the equation of the line of best fit for the inverse relation using technology. Write the inverse relation in function notation.

(e) Determine an equation of a linear model for the inverse relationship using your answer from (b).

(f) Graph the equations that you wrote for (d) and (e) on the same axes. Compare and contrast the graphs. Explain what you notice.

(g) Use a linear model to estimate how much beef will be sold if the price is \$4.75/kg.

(h) Use a linear model to estimate how much beef will be sold if the price is \$5.00/kg. Explain.

(i) Use a linear model to estimate the price the manager should charge if he hopes to sell 4000 kg of beef in one week.

18. A Canadian address can be converted into a six-character postal code, such as N2V 3C2.

(a) Why must this conversion be a function?

(b) Explain why the inverse is not a function.

19. In section 3.2, a binary number was converted into a decimal number. Design a process for the inverse relation, that is, converting a decimal number into a binary number.

20. **Application:** For security, a credit card number is coded in the following way, so that it can be sent as a message. "Subtract each digit from 9."

(a) Code the credit card number 4332 178 256.

(b) A coded credit card number is 207 456 127. What is the original credit card number?

(c) Find $f(x)$ if x represents a single input digit.

(d) Find $f^{-1}(x)$.

(e) Graph the functions $y = f(x)$ and $y = f^{-1}(x)$ on the same axes.

21. To code words as numbers, Watson used this rule:
A is 1, B is 2, C is 3, ... , Z is 26, and a blank is 0. For example, "HI" becomes "89" and "A BALL" becomes "10211212."

(a) Code the word "BOY" using this rule.

(b) Explain why this relation is a function.

(c) Decode "21520201513."

(d) Why is this rule not a good way to encode words?

22. **Check Your Understanding:** In section 3.2, the soft-drink vending machine was an example of a function.

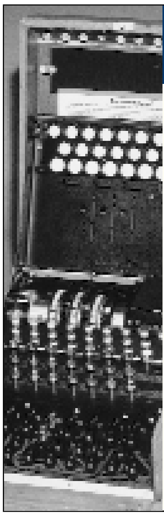
(a) What is the independent variable for the inverse relation?

(b) What is the dependent variable for the inverse relation?

(c) What is the domain of the inverse relation?

(d) What is the range of the inverse relation?

(e) Is the inverse relation a function? Explain.



The Chapter Problem—Cryptography

In this section, you worked with the inverse function. Apply what you learned to answer these questions about the Chapter Problem on page 218.

CP5. Graph the coding function.

CP6. How does this graph show that the relation is a function?

CP7. Is the inverse relation a function? Why must the inverse relation be a function for any encryption method?

CP8. Graph the inverse relation on the same axes in a different colour.

CP9. Use the graph of the inverse relation to decode the message.

CP10. What features of this coding technique make it easy to decode?