Functions: Concept and Notation 3.2

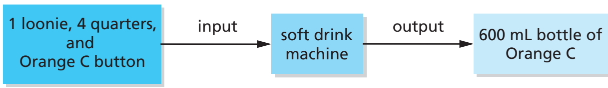
In this section, you will develop the concept of a function. You will also learn how to write a function and how to work with functions.

Part 1: The Function

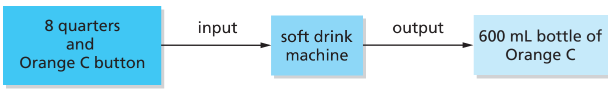
A vending machine behaves like a function. In other words, you get a predictable and reliable output from a vending machine depending on what you put into it.

A soft-drink vending machine allows the customer to put in nickels, dimes, quarters, loonies, or toonies, and then make a choice.

When you input the correct combination of coins and push a button, you expect to get the correct soft drink. The "input" for this function is a number of coins and a button on the machine. The "output" is a soft drink.



A different input might produce the same output. For example:



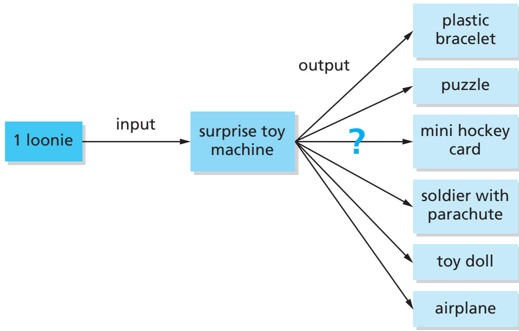
However, you would not expect the same inputs to produce different outputs. For example, suppose that you input two loonies and pushed the Orange C button. Would you expect to get a bottle of Fizz?

The combination of coins and the soft-drink button used are the **independent variables**. The soft drink received is the **dependent variable**. The machine operates as a **function** because the machine produces a guaranteed output for a specific input. In other words, a unique value of the dependent variable is produced for a specific set of values of the independent variables. The soft drink received is a function of the combination of coins and the soft-drink button pushed.

A **function** produces a unique value of the dependent variable for each value of the independent variable.

The set of all possible values of the independent variable is called the **domain** of the function. The set of all possible values of the dependent variable is called the **range** of the function.

There are some vending machines that are not functions. One example is a machine that rewards a child with a surprise toy when the child inserts a loonie.



The prize is not a function of the coin inserted in the machine since there are a variety of toys that the child could receive for $1. One value of the independent variable produces more than one value of the dependent variable.

Example 1

For the soft-drink machine,

(a) what is the domain of the function? (b) what is the range of the function?

Solution

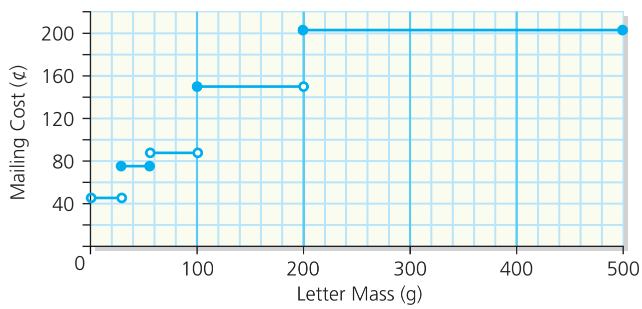
(a) The domain is the set of all possible combinations of coins, which total $2 or $1.25, and a specific button.

(b) The range is the set of all possible types of drinks that the machine contains.

Example 2

The table and the graph show the cost of mailing a first-class letter in Canada in 2001.

Mass of Letter	Mailing Cost
less than 30 g	47¢
30 g to 50 g	75¢
over 50 g but less than 100 g	94¢
over 100 g but less than 200 g	$1.55
200 g to 500 g	$2.05



(a) What is the cost of mailing a letter with mass

i. 20 g? **ii.** 100 g?

(b) State the domain of the relation in set notation.

(c) State the range of the relation.

(d) Why must this relation be a function?

Solution

(a) Estimate each cost using the graph. Find the exact cost in the table.

i. It costs 47¢ to mail a 20-g letter.
ii. It costs $1.55 to mail a 100-g letter.

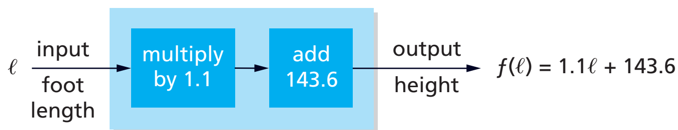
(b) In the graph, the first interval starts at 0 g and the last interval ends with 500 g, so the domain is the set of all real numbers greater than 0 but less than or equal to 500. Use set notation to describe the domain, or range, with symbols. Let m be the mass in grams. Then the domain is $\{m \mid 0 < m \leq 500, m \in \mathbf{R}\}$, which is read, "the set of all values m, such that m is greater than 0 and less than or equal to 500, and m is an element of the set of real numbers."

(c) The range is the set of all possible prices. From the table, the range is {47¢, 75¢, 94¢, $1.55, $2.05}. From the graph, you can only estimate these values.

(d) Consumers would not pay different prices for two letters with the same mass. Each different mass has a unique price.

Part 2: Function Notation

In section 3.1, you determined the line of best fit for data about a group of students in a grade 11 math class. The equation $h = 1.1\ell + 143.6$ is one possible model for the relation between height and foot length for students in this class. Using function notation, you can write this equation as $f(\ell) = 1.1\ell + 143.6$. The name of the function is f, and $f(\ell)$ refers to the output of the function. $f(\ell)$ is the value of the function f at ℓ and is read "f at ℓ" or "f of ℓ."



Example 3

Determine the height of a student whose footprint is 31 cm long.

Solution

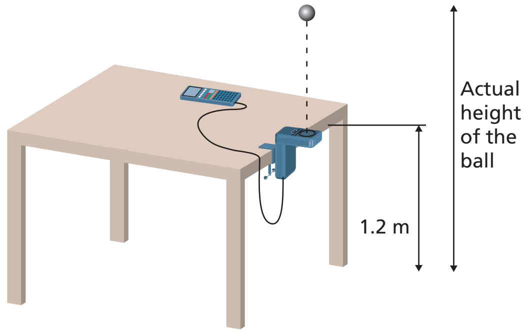
$f(31)$ will be the output of f when the input is 31. To calculate $f(31)$, replace ℓ with 31 and evaluate.

$$f(\ell) = 1.1\ell + 143.6$$
$$f(31) = 1.1(31) + 143.6$$
$$= 177.7$$

The student whose footprint is 31 cm long may be 177.7 cm tall.

Example 4

Peter throws a ball in the air, and Hazel uses a Calculator-Based Ranger (CBR) to record the height of the ball as it goes up and then down. They use quadratic regression to find the equation of the curve of best fit, $f(t) = -4.9t^2 + 1.9t + 1.1$, where $f(t)$ is the height in metres after t seconds. The CBR was 1.2 m above the ground during the experiment. Hazel didn't start recording until 0.3 s after Peter released the ball.



(a) Calculate the height at which Peter releases the ball.

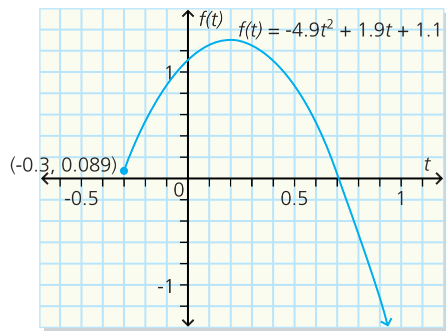
(b) Calculate the height of the ball after 1 s.

(c) Rewrite the function so that the height of the ball above the ground is a function of the time since Peter released the ball.

(d) Use the new function to verify your answers from (a) and (b).

Solution

(a) The graph of f shows the height of the ball. Note that the CBR started recording data 0.3 s after Peter threw the ball. When $t = 0$, the ball had already been in the air for 0.3 s. To calculate the height of the ball when it was released, find $f(-0.3)$.



$$f(-0.3) = -4.9(-0.3)^2 + 1.9(-0.3) + 1.1$$
$$= -4.9(0.09) - 0.57 + 1.1$$
$$= 0.089$$

Peter released the ball 0.089 m above the CBR, which is 1.2 m above the floor.

$$0.089 + 1.2 = 1.289$$

When Peter released the ball, it was about 1.3 m above the floor.

(b) The CBR started recording data 0.3 s after Peter threw the ball. Since $1 - 0.3 = 0.7$, use $t = 0.7$ s. The total height of the ball after 1 s is the sum of $f(0.7)$ and 1.2 m, the height at which the CBR is attached to the table.

$$\begin{aligned} f(0.7) + 1.2 &= -4.9(0.7)^2 + 1.9(0.7) + 1.1 + 1.2 \\ &= -4.9(0.49) + 1.33 + 2.3 \\ &= 1.229 \end{aligned}$$

After 1 s, the ball is about 1.2 m above the ground.

(c) Let $h(t)$ be the height of the ball in metres t seconds after Peter releases the ball. Subtract 0.3 from the value of t in $f(t)$ and add 1.2 to the height.

$$\begin{aligned} h(t) &= f(t - 0.3) + 1.2 \\ &= -4.9(t - 0.3)^2 + 1.9(t - 0.3) + 1.1 + 1.2 \\ &= -4.9(t^2 - 0.6t + 0.09) + 1.9t - 0.57 + 2.3 \\ &= -4.9t^2 + 2.94t - 0.441 + 1.9t + 1.73 \\ &= -4.9t^2 + 4.84t + 1.289 \end{aligned}$$

(d) For (a), $t = 0$ when Peter released the ball.

$$\begin{aligned} h(0) &= -4.9(0)^2 + 4.84(0) + 1.289 \\ &= 1.289 \end{aligned}$$

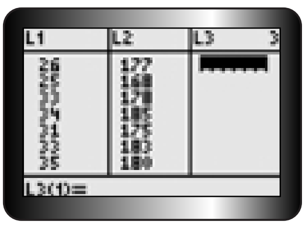
For (b), $t = 1$ when the ball is released after 1 s.

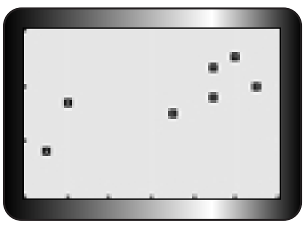
$$\begin{aligned} h(1) &= -4.9(1)^2 + 4.84(1) + 1.289 \\ &= 1.229 \end{aligned}$$

The answers obtained using $h(t)$ are the same as those for (a) and (b).

Part 3: The Vertical Line Test

In section 3.1, you investigated the relationship between the footprint length and height. All of the seven students had footprints of different lengths. Al remeasures the footprint that is 33.5 cm long and finds that it is actually 33 cm long. Here are the revised lists and a new scatter plot.





Think, Do, Discuss

1. (a) Why is it no longer possible to identify every person in the group only by the length of his or her footprint?

(b) How do you know this relation is not a function?

2. (a) Place a ruler on the vertical axis (the height axis). Move the ruler slowly to the right. As you move the ruler to the right, note the number of points on the graph of the relation that lie along the edge of the ruler at any one time.
   (b) Which two points lie on the same line? Explain why these two points show that this relation is not a function.
   (c) Draw a line through these two points. Describe the direction of this line.
   (d) In your own words, describe a test to determine whether a relation is a function using its graph.

Consolidate Your Understanding

1. How can you tell if a relation is a function?
2. Why is it useful when a relation is a function?
3. How can you determine the domain and range of a function from its graph?
4. A function is given by $f(x) = 3x - 5$. Describe this function as a rule.
5. What does $f(2)$ mean?
6. How can you tell from the graph of a relation whether it is a function or not? Explain.

Focus 3.2

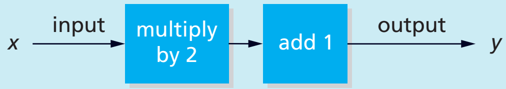
Key Ideas

- The set of all possible input values of a relation is called the **domain**. The set of all possible output values is called the **range**.
- A function is a relation in which each element of the domain corresponds to exactly one element of the range.
- You may describe a relation and a function as
  - a set of ordered pairs. For example, $\{(0, 1), (3, 4), (2, -5)\}$.
  - a table. For example,

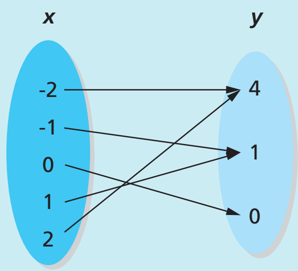
x	y
1	5
2	7
3	9

  - a description in words. For example, there is a relationship between the age and the height of students in your class.
  - a rule. For example, the output is 4 more than the input.
  - an equation. For example, $y = 2x + 1$.

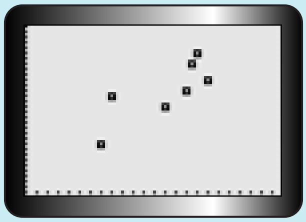
- an input/output diagram. For example,



- function notation. For example, $f(x) = 2x + 1$.
- an arrow diagram. For example,



- a graph or a scatter plot. For example,



- $f(x)$ is called function notation and is read "f at x" or "f of x."
- $f(x)$ represents an expression for determining the value of the function f for any value of x.
- $f(3)$ represents the value of the function (the output) when x (the input) is 3. $f(3)$ is the y-coordinate of the point on the graph of f with x-coordinate 3.
- y usually represents the output. Then $y = f(x)$ is the equation of a function f.
- The **vertical line test** can be used to determine if the graph of a relation is a function.

  The relation is not a function if you can draw a vertical line through two or more points on the graph of the relation. The relation is a function if you cannot draw a vertical line through two or more points on the graph.

Example 5

How old are you right now?

(a) Would the answer to this question produce a function? Explain.

(b) Let the domain be the set of all the students in your class. Then what is the range?

Solution

(a) Yes, because every person you ask has only one possible age.

(b) The range is probably {15, 16, 17}.

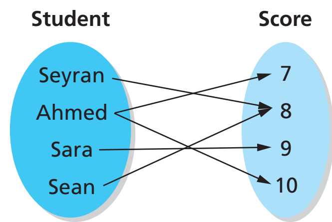
Example 6

A relation is shown in the arrow diagram. The input is a student's name and the output is a score out of 10 on a math quiz. For example, Seyran scored 8 out of 10 on the math quiz.

(a) Write this relation as a set of ordered pairs.

(b) State the domain of this relation.

(c) Is this relation a function? Explain.



Solution

(a) {(Seyran, 8), (Ahmed, 7), (Ahmed, 10), (Sara, 9), (Sean, 8)}.

(b) The domain is {Seyran, Ahmed, Sara, Sean}.

(c) The arrow diagram shows that Ahmed had two scores, because he redid the math quiz after he scored a low mark the first time. So there are two values in the range for one value in the domain. This relation is not a function.

Example 7

The equation $y = 4x + 1$ describes a relation.

(a) The input for this relation is any real number. Describe a rule for calculating the output value.

(b) Graph this relation.

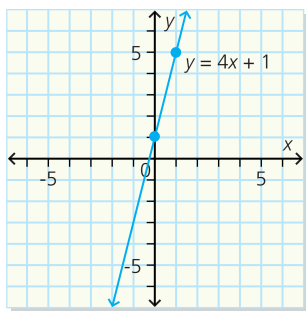
(c) Is this relation a function? Explain.

Solution

(a) The equation shows that x is the input and $4x + 1$ is the output. Multiply the input by 4, and then add 1 to get the output.

(b) This relation is a line with slope 4 and y-intercept 1.

(c) Each value of x yields only one value of y, so this relation is a function. Verify using the vertical line test.



Example 8

For the function $g(x) = x^2 - 4x$,

(a) evaluate $g(-3)$

(b) simplify $g(a + 1)$

(c) graph g

(d) find the domain and range of g

Solution

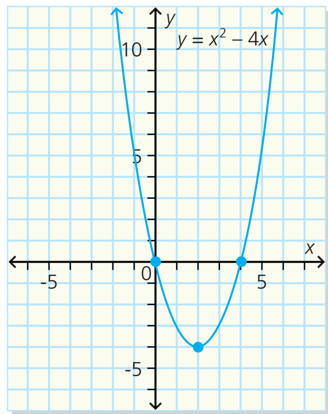
(a)
$$\begin{aligned} g(-3) &= (-3)^2 - 4(-3) \\ &= 9 + 12 \\ &= 21 \end{aligned}$$

(b)
$$\begin{aligned} g(a + 1) &= (a + 1)^2 - 4(a + 1) \\ &= a^2 + 2a + 1 - 4a - 4 \\ &= a^2 - 2a - 3 \end{aligned}$$

(c) Express the equation in vertex form by completing the square.

$$\begin{aligned} g(x) &= x^2 - 4x \\ &= x^2 - 4x + 4 - 4 \\ &= (x - 2)^2 - 4 \end{aligned}$$

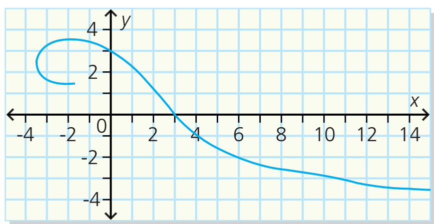
The vertex is $(2, -4)$, and the parabola opens up. Factor the original function to get $g(x) = x(x - 4)$. The x-intercepts are 0 and -4.



(d) Any real number may be input for g, so the domain is **R**. The value of y is not less than -4, so the range is $\{y \mid y \geq -4, y \in \mathbf{R}\}$.

Example 9

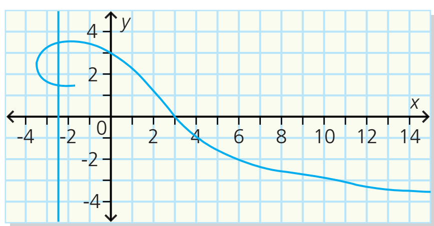
The graph represents an infinite number of points that belong to a relation.



Use the vertical line test to explain why this relation is not a function.

Solution

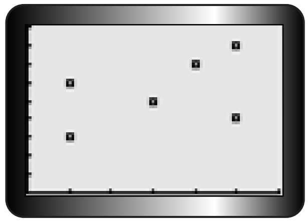
At $x = -2.5$, the vertical line passes through two values of y, about 1.5 and 3.2, so this relation is not a function.



Practise, Apply, Solve 3.2

A

1. The scatter plot shows a relation. The marks on each axis indicate single units.



(a) State the domain and range of this relation.

(b) Draw an arrow diagram to illustrate the relation.

(c) Is this relation a function? Explain.

2. For each of the following, state

i. the domain and range

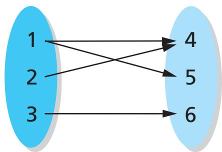
ii. whether it defines a function or not, and justify your answer

(a) {(1, 2), (3, 1), (4, 2), (7, 2)}

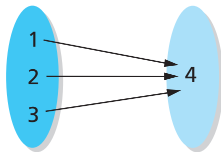
(b) {(1, 2), (1, 3), (4, 5), (6, 1)}

(c) {(1, 0), (0, 1), (2, 3), (3, 2)}

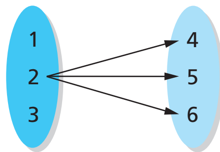
(d)



(e)



(f)



3. Consider the $\sqrt{\ }$ key on your calculator. Recall that $\sqrt{\ }$ means the positive square root.

(a) What is the output if the input is 25?

(b) Does the output have more than one value for any input value?

(c) Why must this operation be a function? Explain.

(d) Are there any numbers that cannot be used as input?

(e) State the domain of this function.

4. Consider the rule "Take the square root of the input number to get the output number."

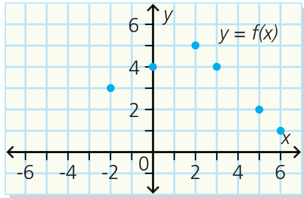
(a) What is the output if the input is 25?

(b) Does the output have more than one value for any input value?

(c) Is this relation a function? Explain.

(d) Are there any numbers that cannot be used as input?

5. The graph of $y = f(x)$ is shown.



i. State the domain and range of f.

ii. Evaluate.

(a) $f(3)$ (b) $f(5)$

(c) $f(5 - 3)$ (d) $f(5) - f(3)$

iii. In part ii, why is the function in (d) not the same as the answer in (c)?

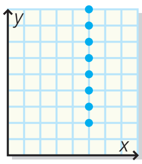
iv. $f(2) = 5$. What is the corresponding ordered pair? What does 2 represent? What does $f(2)$ represent?

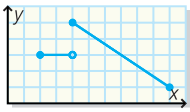
6. For $g(x) = 3 - 2x$, find

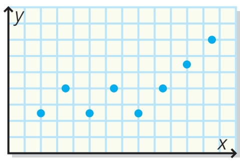
(a) $g(3)$ (b) $g(0)$ (c) $g(-2)$ (d) $2g(1)$

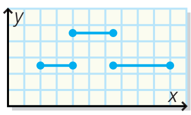
(e) $g(-2) - 3$ (f) $g(-2 - 3)$ (g) $g(a)$

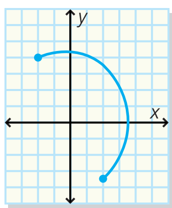
7. State whether each graph shows a function. Justify your answer.

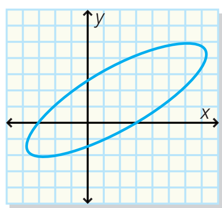
(a)


(b)


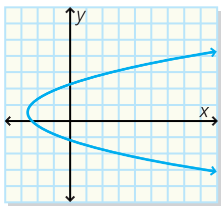
(c)


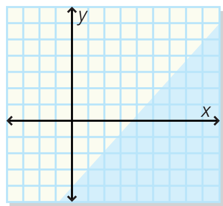
(d)


(e)


(f)


(g)


(h)


(i)


8. Consider the function $g(t) = 3t + 5$.

i. Create a table and graph the function.

ii. Determine each value.

(a) $g(0)$ (b) $g(1)$ (c) $g(2)$

(d) $g(3)$ (e) $g(1) - g(0)$ (f) $g(2) - g(1)$

(g) $g(3) - g(2)$ (h) $g(1001) - g(1000)$ (i) $g(a + 1) - g(a)$

(j) $\frac{g(4) - g(0)}{4 - 0}$

iii. In part ii, what are the answers to (e), (f), and (g), as a group, commonly called? Why is the answer to (j) the same as those for (e), (f), (g), (h), and (i)?

9. **Knowledge and Understanding:** For each of the following,

i. sketch the relation
ii. state the domain and range
iii. explain why the relation is a function

(a) $y = \frac{1}{x}$ (b) $y = 6 - 2x$

(c) $x + 3y = 6$ (d) $y = x^2 + 3$

10. Bill called a garage to ask for a price quote on tires. Bill told the clerk what size of tire he needed, and the clerk told him the price. When Bill called back later that same day, another clerk told him a different price. How could this happen? Explain why this situation is not a function. How could you adapt this situation to make it a function?

11. The adjacent table lists all of the ordered pairs belonging to a function g.

i. Determine the equation of the line that passes through these points.
ii. Write $g(x)$.
iii. Evaluate.

(a) $g(5)$ (b) $g(5 - 3)$

(c) $g(5) - g(3)$ (d) $2g(3) - 5$

x	y
1	5
2	7
3	9
4	11
5	13

12. Consider the function $f(s) = s^2 - 6s + 9$.

i. Create a table for the function.
ii. Determine each value.

(a) $f(0)$
(b) $f(1)$
(c) $f(2)$
(d) $f(3)$
(e) $f(4)$
(f) $f(1) - f(0)$
(g) $f(2) - f(1)$
(h) $f(3) - f(2)$
(i) $f(4) - f(3)$
(j) $[f(2) - f(1)] - [f(1) - f(0)]$
(k) $[f(3) - f(2)] - [f(2) - f(1)]$
(l) $[f(4) - f(3)] - [f(3) - f(2)]$
(m) $[f(1002) - f(1001)] - [f(1001) - f(1000)]$

iii. In part ii, what are the answers to (f), (g), (h), and (i), as a group, commonly called? What are the answers to (j), (k), and (l), as a group, commonly called?

13. The graph shows $f(x) = 2(x - 3)^2 - 1$.

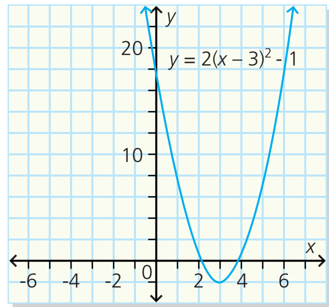
(a) Evaluate $f(-2)$.

(b) What does $f(-2)$ represent on the graph of f?

(c) State the domain and range of the relation.

(d) How do you know that f is a function from its graph?

(e) How do you know that f is a function from its equation?



14. Consider the relation $y = x^2 - 3x$.

(a) Are there any values of the independent variable for which the dependent variable is not unique?

(b) Is this relation a function? Explain.

15. A relation is defined by $x^2 + y^2 = 25$.

(a) Sketch a graph of the relation.

(b) Is this relation a function? Explain.

16. For each of the following,

i. graph the relation

ii. state the domain and range

iii. is the relation a function? Why or why not?

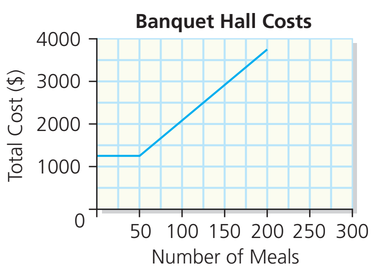
(a) $y = 3x - 1$ (b) $y = 10 - 4.9x^2$ (c) $y = 3(x - 2)^2 - 5$

(d) $y = \frac{1}{x^2}$ (e) $x^2 - y = 3x$ (f) $y = x(x - 4)$

(g) $5x + 3y = 15$

17. State the domain and range of the function $y = \sqrt{x - 1} + 2$.

18. The cost of renting a banquet hall depends on the size of the room and the number of meals served. A graph of the number of meals versus cost is shown.



(a) What problems would the banquet hall have if this relation were not a function?

(b) What is the domain and range of this function?

(c) Why does the domain have an upper limit?

(d) Why is the graph a reasonable representation of the cost to rent a banquet hall?

19. **Communication:** A ball is thrown, and its height is recorded a number of different times. The ball reaches a maximum height of 20 m after 1.5 s.

(a) Sketch the relation. Time is the independent variable and height is the dependent variable.

(b) State a reasonable domain and range for the relation.

(c) Is this relation a function? Explain.

(d) A student decided to use height as the independent variable and time as the dependent variable. Sketch a graph of this relation.

(e) Why is this relation not a function?

20. A freight delivery company charges \$4/kg for any order less than 100 kg and \$3.50/kg for any order of at least 100 kg.

(a) Why must this relation be a function?

(b) What is the domain of this function? What is its range?

(c) Graph this function.

(d) What suggestions can you offer to the company for a better pricing structure? Support your answer.

21. **Thinking, Inquiry, Problem Solving:** A women's clothing store owner wants to create a payment plan to motivate employees to sell as much as possible. However, the owner wants to give the employees some security during slow times when there are few customers. She has three possible plans as follows:

Plan A: a flat salary of \$300/week

Plan B: a payment of \$200/week, plus 5% commission on sales

Plan C: 7.5% commission on sales

(a) What is a reasonable domain for this relation? What is a reasonable range?

(b) Why is it important to the employees for this relation to be a function?

(c) Create a plan that combines these payment plans for the best results. Justify your decisions.

22. **Application:** The amount owed on a loan depends on several variables. Assume that the interest is compounded annually.

(a) Graph the amount owed versus the interest rate for a loan of \$10 000 over five years.

(b) Describe the domain and range of this function.

23. Sara asked each of her family members to measure his or her foot length. Then she graphed the relationship between foot length and age, using age as the independent variable.

(a) Will this relationship be a function? Explain.

(b) Sara asked all of her friends to measure the foot lengths of their family members. Then she combined the data. Will this relationship be a function? Explain.

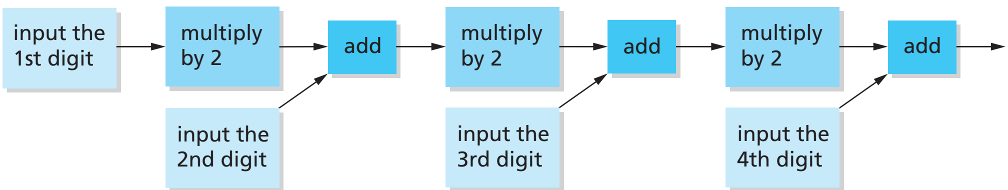
(c) Sara modelled the relationship with a line of best fit. Is this model a function? Explain.

(d) Is foot length a function of age? Explain.

24. **Check Your Understanding:** Create your own examples of a relation that is not a function and another that is a function, using descriptions, graphs, tables, or equations. Explain the difference between the relation and the function.

C

25. Binary numbers are used by computers and other digital devices. A binary number has only 0s and 1s. The binary number 1101 is equivalent to the decimal number 13. To convert a binary number to a decimal number, use the following process. Start at the left.

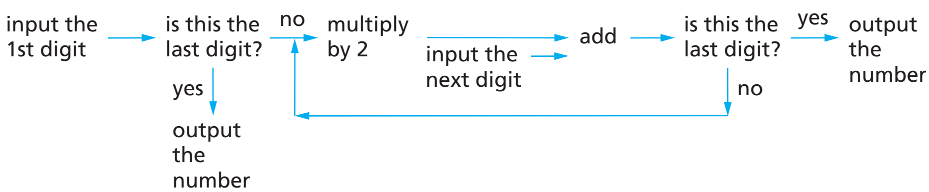


(a) Use this process to decode the binary number 1101.

(b) Why is this process a function?

(c) The diagram below shows how to convert any binary number into a decimal number. Use this process to convert each of the following into a decimal number.

i. 10101 **ii.** 11 **iii.** 1000



(d) Suppose the domain of this function is the set of all one-, two-, three-, or four-digit binary numbers. What is the range of the function?

(e) How many binary digits, or bits, are needed to represent all of the letters and symbols on a keyboard?

26. Many calculators can generate a random number. Using a TI-83 Plus calculator, press [MATH] and scroll right to **PRB**. Select **1:rand** by pressing [ENTER]. Press [ENTER] again to get a random number. To get another random number, you can repeat these steps or just press [ENTER] to repeat the command.

(a) Generate five random numbers in a row and record these in a list.

(b) Why does it appear that **rand** is not a function? Why is this not a problem?

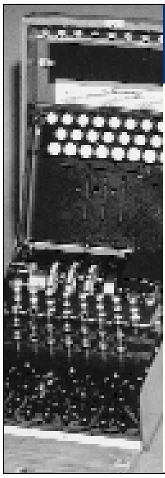
(c) It is also possible to store numbers in **rand**. To store the value 0, press [0] [STO▸] [MATH]. Scroll right to **PRB** and select **1:rand** by pressing [ENTER]. Now generate and write another list of random numbers. Compare your list to those of other students. What do you notice?

(d) Enter the first number from your list in (a) and store it in **rand**. Generate and write another list of random numbers and compare them to the numbers in your first list.

(e) Generate several lists of three random numbers by storing 0.4, 0.8, 1.2, 1.6, 2.0, 2.4, and 2.8 in **rand**. What do you notice?

(f) Describe how your random number generator seems to work.

(g) Is **rand** a function? Explain. Why might this be a problem?



The Chapter Problem—Cryptography

In this section, you were introduced to the idea of a function. Apply what you have learned to answer these questions about the Chapter Problem on page 218.

CP1. The coding method described in the Chapter Problem is an example of a function. Why?

CP2. What is the domain of this function? Why?

CP3. What is the range of this function? Why?

CP4. What does it mean "to break the code"?