Topic 2—Functions and equations

24 hours

The aims of this topic are to explore the notion of a function as a unifying theme in mathematics, and to apply functional methods to a variety of mathematical situations. It is expected that extensive use will be made of technology in both the development and the application of this topic, rather than elaborate analytical techniques. On examination papers, questions may be set requiring the graphing of functions that do not explicitly appear on the syllabus, and students may need to choose the appropriate viewing window. For those functions explicitly mentioned, questions may also be set on composition of these functions with the linear function $y = ax + b$.

	Content	Further guidance	Links
2.1	Concept of function $f : x \mapsto f(x)$. Domain, range; image (value). Composite functions. Identity function. Inverse function f^{-1}. **Not required:** domain restriction.	*Example*: for $x \mapsto \sqrt{2-x}$, domain is $x \le 2$, range is $y \ge 0$. A graph is helpful in visualizing the range. $(f \circ g)(x) = f(g(x))$. $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$. On examination papers, students will only be asked to find the inverse of a *one-to-one* function.	**Int:** The development of functions, Rene Descartes (France), Gottfried Wilhelm Leibniz (Germany) and Leonhard Euler (Switzerland). **TOK:** Is zero the same as “nothing”? **TOK:** Is mathematics a formal language?
2.2	The graph of a function; its equation $y = f(x)$. Function graphing skills. Investigation of key features of graphs, such as maximum and minimum values, intercepts, horizontal and vertical asymptotes, symmetry, and consideration of domain and range. Use of technology to graph a variety of functions, including ones not specifically mentioned. The graph of $y = f^{-1}(x)$ as the reflection in the line $y = x$ of the graph of $y = f(x)$.	**Note** the difference in the command terms “draw” and “sketch”. An analytic approach is also expected for simple functions, including all those listed under topic 2. Link to 6.3, local maximum and minimum points.	**Appl:** Chemistry 11.3.1 (sketching and interpreting graphs); geographic skills. **TOK:** How accurate is a visual representation of a mathematical concept? (Limits of graphs in delivering information about functions and phenomena in general, relevance of modes of representation.)

	Content	Further guidance	Links
2.3	Transformations of graphs. Translations: $y = f(x) + b$; $y = f(x - a)$. Reflections (in both axes): $y = -f(x)$; $y = f(-x)$. Vertical stretch with scale factor p: $y = pf(x)$. Stretch in the x-direction with scale factor $\frac{1}{q}$: $y = f(qx)$. Composite transformations.	Technology should be used to investigate these transformations. Translation by the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ denotes horizontal shift of 3 units to the right, and vertical shift of 2 down. *Example*: $y = x^2$ used to obtain $y = 3x^2 + 2$ by a stretch of scale factor 3 in the y-direction followed by a translation of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.	**Appl:** Economics 1.1 (shifting of supply and demand curves).
2.4	The quadratic function $x \mapsto ax^2 + bx + c$: its graph, y-intercept $(0, c)$. Axis of symmetry. The form $x \mapsto a(x - p)(x - q)$, x-intercepts $(p, 0)$ and $(q, 0)$. The form $x \mapsto a(x - h)^2 + k$, vertex (h, k).	Candidates are expected to be able to change from one form to another. Links to 2.3, transformations; 2.7, quadratic equations.	**Appl:** Chemistry 17.2 (equilibrium law). **Appl:** Physics 2.1 (kinematics). **Appl:** Physics 4.2 (simple harmonic motion). **Appl:** Physics 9.1 (HL only) (projectile motion).

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2.5	The reciprocal function $x \mapsto \frac{1}{x}$, $x \neq 0$: its graph and self-inverse nature. The rational function $x \mapsto \frac{ax+b}{cx+d}$ and its graph. Vertical and horizontal asymptotes.	*Examples*: $h(x) = \frac{4}{3x-2}$, $x \neq \frac{2}{3}$; $y = \frac{x+7}{2x-5}$, $x \neq \frac{5}{2}$. Diagrams should include all asymptotes and intercepts.	
2.6	Exponential functions and their graphs: $x \mapsto a^x$, $a > 0$, $x \mapsto \mathrm{e}^x$. Logarithmic functions and their graphs: $x \mapsto \log_a x$, $x > 0$, $x \mapsto \ln x$, $x > 0$. Relationships between these functions: $a^x = \mathrm{e}^{x \ln a}$; $\log_a a^x = x$; $a^{\log_a x} = x$, $x > 0$.	Links to 1.1, geometric sequences; 1.2, laws of exponents and logarithms; 2.1, inverse functions; 2.2, graphs of inverses; and 6.1, limits.	**Int:** The Babylonian method of multiplication: $ab = \frac{(a+b)^2 - a^2 - b^2}{2}$. Sulba Sutras in ancient India and the Bakhshali Manuscript contained an algebraic formula for solving quadratic equations.

	Content	Further guidance	Links
2.7	Solving equations, both graphically and analytically.	Solutions may be referred to as roots of equations or zeros of functions.	
	Use of technology to solve a variety of equations, including those where there is no appropriate analytic approach.	Links to 2.2, function graphing skills; and 2.3–2.6, equations involving specific functions. *Examples*: $\mathrm{e}^x = \sin x$, $x^4 + 5x - 6 = 0$.	
	Solving $ax^2 + bx + c = 0$, $a \neq 0$. The quadratic formula.		
	The discriminant $\Delta = b^2 - 4ac$ and the nature of the roots, that is, two distinct real roots, two equal real roots, no real roots.	*Example*: Find k given that the equation $3kx^2 + 2x + k = 0$ has two equal real roots.	
	Solving exponential equations.	*Examples*: $2^{x-1} = 10$, $\left(\frac{1}{3}\right)^x = 9^{x+1}$. Link to 1.2, exponents and logarithms.	
2.8	Applications of graphing skills and solving equations that relate to real-life situations.	Link to 1.1, geometric series.	**Appl:** Compound interest, growth and decay; projectile motion; braking distance; electrical circuits. **Appl:** Physics 7.2.7–7.2.9, 13.2.5, 13.2.6, 13.2.8 (radioactive decay and half-life)