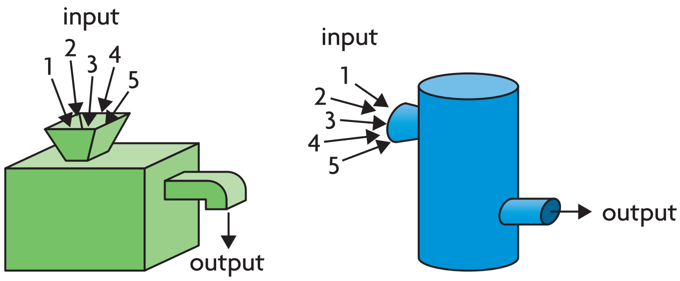
4.3 Equation-Solving Strategies

GOAL

Solve equations where the variable appears on both sides of the equality.

LEARN ABOUT the Math



machine 1: $\frac{1}{2}x + \frac{3}{4}$ machine 2: $x - \frac{1}{2}$

Atish and Sara are playing a game on their computers. The goal of the game is to be the first to figure out which input will make the two machines generate the same output.

They have input several values. Here are their results:

	Machine 1	**Machine 2**
Input	Output	Output
0	0.75	−0.5
2	1.75	1.5
4	2.75	3.5
6	3.75	5.5

? What strategies can they use to win the game quickly?

EXAMPLE 1 Selecting a strategy to solve an equation

Determine the input value that will result in the same output value for both machines.

Klint's Solution: Using a guess-and-test strategy

Input Value	Output Value	
x	Machine 1 $\frac{1}{2}x + \frac{3}{4}$	Machine 2 $x - \frac{1}{2}$
0	0.75	−0.5
2	1.75	1.5
4	2.75	3.5

I noticed that when the input is 2, the output value of Machine 1 is greater than the output value of Machine 2. But, when the input is 4, the reverse is true.

$x = 3$

I thought the answer must be between 2 and 4, so I tried 3.

Input Value	Output Value	
x	Machine 1 $\frac{1}{2}x + \frac{3}{4}$	Machine 2 $x - \frac{1}{2}$
0	0.75	−0.5
2	1.75	1.5
3	2.25	2.5

The output value of Machine 1 is still greater when $x = 2$ and less when $x = 3$. The solution must be between 2 and 3.

$x = 2\frac{1}{2}$

I chose an input value of $2\frac{1}{2}$, which is between 2 and 3. I tested this value in each machine.

$x = \frac{5}{2}$

I rewrote the fraction as an improper fraction, and then, calculated the outputs.

Input Value	Output Value	
x	Machine 1 $\frac{1}{2}x + \frac{3}{4}$	Machine 2 $x - \frac{1}{2}$
0	0.75	−0.5
2	1.75	1.5
2.5	2.00	2.0

When the input is $2\frac{1}{2}$, or 2.5, both machines have an output value of 2.

An input of $2\frac{1}{2}$ will produce equal outputs on both machines.

When an equation contains fractions, it is often easier to solve if you can rewrite it as an equivalent equation that does not contain fractions. This can be done using a common denominator. You can then use inverse operations to solve the resulting equation.

Dion's Solution: Using a common denominator and inverse operations to isolate the variable

$$\frac{1}{2}x + \frac{3}{4} = x - \frac{1}{2}$$

I created an equation to figure out what value of *x* would give the same output values

$$4\left(\frac{1}{2}x + \frac{3}{4}\right) = 4\left(x - \frac{1}{2}\right)$$

I tried to write an equivalent equation that didn't contain fractions. I knew that if I performed the same operation on both sides of the equation, the result is an equivalent equation. I decided to multiply both sides by 4, since both denominators will divide into this number, eliminating the fractions. This is also the lowest common denominator between 2 and 4.

$$4\left(\frac{1}{2}x\right) + 4\left(\frac{3}{4}\right) = 4(x) - 4\left(\frac{1}{2}\right)$$

$$\frac{4}{2}x + \frac{12}{4}x = 4x - \frac{4}{2}$$

$$2x + 3 = 4x - 2$$

I used the distributive property to expand, and then, simplified by multiplying.

$$2x + 3 - 3 = 4x - 2 - 3$$

$$2x = 4x - 5$$

I used inverse operations to solve the equation. I decided to undo $+ 3$ by subtracting 3 from both sides.

$$2x + 5 = 4x - 5 + 5$$

$$2x + 5 = 4x$$

To undo $- 5$, I added 5 to both sides.

$$2x + 5 - 2x = 4x - 2x$$

$$5 = 2x$$

To isolate the variable term I had to undo $+ 2x$. I subtracted 2x from both sides

$$\frac{5}{2} = \frac{2x}{2}$$

$$\frac{5}{2} = x$$

To solve for *x*, I used the inverse operation of multiply by 2 and divided both sides by 2.

$$2\frac{1}{2} = x$$

I rewrote the fraction as a mixed number.

The input that will produce the same output on both machines is $2\frac{1}{2}$.

Check:

When $x = \frac{5}{2}$,

Left Side	Right Side
$\frac{1}{2}x + \frac{3}{4}$	$x - \frac{1}{2}$
$= \frac{1}{2}\left(\frac{5}{2}\right) + \frac{3}{4}$	$= \frac{5}{2} - \frac{1}{2}$
$= \frac{5}{4} + \frac{3}{4}$	$= \frac{4}{2}$
$= \frac{8}{4}$	$= 2$
$= 2$	

I checked the solution by entering the input value into the left hand and right hand sides separately.

Since the left side and right side both resulted in the same output, I knew my solution was correct.

Reflecting

A. How does solving the equation $\frac{1}{2}x + \frac{3}{4} = x - \frac{1}{2}$ show that there is only one value of x for which the machines have the same output?

B. Why does Klint's strategy only provide an estimate of the solution, while Dion's provides an exact solution?

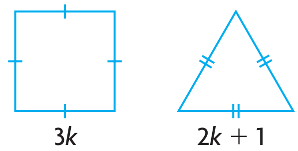
C. How did Dion decide what number to multiply both sides of the equation by to eliminate the fractions?

D. How did Dion know which inverse operations to use in order to group the variables on one side of the equation and the constant terms on the other side?

APPLY the Math

EXAMPLE 2 Using an inverse operations strategy to solve a problem represented by an equation

The square and equilateral triangle shown have the same perimeters. What are the dimensions of each figure?



Kayla's Solution

$$4(3k) = 3(2k + 1)$$

I used the expression $4 \times 3k$ to calculate the perimeter of the square and $3 \times (2k + 1)$ to calculate the perimeter of the triangle.
I made these two expressions equal because the perimeters are the same.

$$12k = 6k + 3$$

I used the distributive property on the right side of the equation to simplify it.

$$12k - 6k = 6k + 3 - 6k$$
$$6k = 3$$
$$\frac{6k}{6} = \frac{3}{6}$$
$$k = \frac{1}{2}$$

I used inverse operations to group the variables on the left side of the equation. I chose the left because $12k$ is larger than $6k$ and my variable would have a positive coefficient.

When $k = \frac{1}{2}$,
Perimeter of the square:
$4(3k)$
$= 4\left(3 \times \frac{1}{2}\right)$

I checked my solution by finding the perimeter of the square and the triangle when $k = \frac{1}{2}$.

$= 6$

Since the perimeters of both shapes are 6 units when $k = \frac{1}{2}$, I knew my solution was correct.

Perimeter of the triangle:
$3(2k + 1)$
$= 3\left(2 \times \frac{1}{2} + 1\right)$
$= 6$

When $k = \frac{1}{2}$,

Dimensions of the square:

$3k$

$= \frac{3}{2}$

$= 1\frac{1}{2}$

To find the dimensions, I substituted $k = \frac{1}{2}$ into the expression for the side lengths of the triangle and of the square.

Dimensions of the triangle:

$2k + 1$

$= 2\left(\frac{1}{2}\right) + 1$

$= 1 + 1$

$= 2$

The perimeters are the same when the square has a side length of $1\frac{1}{2}$ and the triangle has a side length of 2.

EXAMPLE 3 Using an equation and an inverse operations strategy to solve a problem

It takes Ryan 2 h to mow the lawn and water the garden. It takes Maria 3 h to do the same. How long would it take them if they worked together?

Abby's Solution

Let x represent the amount of time needed to complete the tasks if Ryan and Maria work together.

I used a variable to represent the time needed to complete the job if they worked together.

Fraction of work done by Ryan: $\frac{x}{2}$

Fraction of work done by Maria: $\frac{x}{3}$

I wrote an expression to represent the fraction of work done by each person:

fraction of work done $= \frac{\text{time actually spent}}{\text{time to do the whole job alone}}$

$$\frac{x}{2} + \frac{x}{3} = 1$$

Since both people are working together, the total of the fractions of work done must be 1. I used this to write an equation.

$$6\left(\frac{x}{2}\right) + 6\left(\frac{x}{3}\right) = 6(1)$$

I wanted to create an equivalent equation without fractions. I knew that if I multiplied both sides of the equation by 2, the $\frac{x}{2}$ would become $1x$, or just x. But I would still have the thirds on this side of the equation. So, I decided to multiply both sides by 2, and then, by 3. That's the same as multiplying by 6. I had to remember to multiply every term and not just the fraction terms or the new equation would not be equivalent.

$$\frac{6x}{2} + \frac{6x}{3} = 6$$

$$3x + 2x = 6$$

$$5x = 6$$

I multiplied, and then, simplified.

$$\frac{5x}{5} = \frac{6}{5}$$

I divided both sides by 5 to isolate the variable.

$$x = 1.2$$

It would take Ryan and Maria 1.2 h to mow the lawn and water the garden if they worked together.

Check:

Left Side	Right Side
	1

I used my calculator to check my solution. Since the left side equals the right side when $x = 1.2$, I know my solution is correct.

In Summary

Key Ideas

- When you solve an equation in which the variable appears on both sides of the equal sign ($ax + b = cx + d$), you can use inverse operations to group the variable terms on one side of the equation.
- If an equation has fraction coefficients and constants, you can use a common denominator to write an equivalent equation with integer coefficients. To do this, multiply each term of the equation by the common denominator. Using the lowest common denominator keeps the numbers in the new equivalent equation as small as possible.

Need To Know

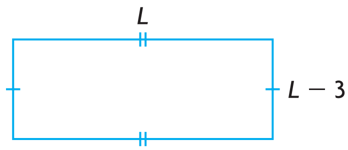
- Sometimes you have to use the distributive property to expand expressions involving brackets before you can collect like terms on each side of the equation before solving for the variable.
- You can check your solution to an equation of the form $ax + b = cx + d$ by substituting the value in each side of the equation and calculating the result. If you get the same value on both sides, then your result is correct.

CHECK Your Understanding

1. Use inverse operations to solve $2x + 4 = 4x - 2$.

2. To determine the dimensions of the rectangle with
   - perimeter 44 cm and
   - width 3 cm less than the length,

   Florence drew a diagram:



   a) Why is it reasonable for Florence to label the width "$L - 3$"?
   b) Create and solve the equation to determine the length of one side.
   c) What are the dimensions of the rectangle?

PRACTISING

3. Given each solved equation below, explain the mathematical reasoning for each step.

   a)
$$\begin{aligned} 2x + 8 &= 4x - 18 & \\ 2x + 8 - 2x &= 4x - 18 - 2x & \text{Step A} \\ 8 &= 2x - 18 & \\ 8 + 18 &= 2x - 18 + 18 & \text{Step B} \\ 26 &= 2x & \\ \frac{26}{2} &= \frac{2x}{2} & \text{Step C} \\ 13 &= x & \end{aligned}$$

   b)
$$\begin{aligned} \frac{1}{2}x + \frac{2}{3} &= 5 & \\ 6 \times \left(\frac{1}{2}x + \frac{2}{3}\right) &= 5 \times 6 & \text{Step A} \\ 3x + 4 &= 30 & \text{Step B} \\ 3x + 4 - 4 &= 30 - 4 & \text{Step C} \\ 3x &= 26 & \\ \frac{3x}{3} &= \frac{26}{3} & \text{Step D} \\ x &= 8\frac{2}{3} & \text{Step E} \end{aligned}$$

4. Explain why the equations in each group are equivalent equations.

a) $5x + 8 = 2(2x - 3)$, $5x + 8 = 4x - 6$, and $5x - 4x = -6 - 8$

b) $\frac{x}{4} + 5 = \frac{1}{3}$, $\frac{3x}{12} + \frac{60}{12} = \frac{4}{12}$, and $3x + 60 = 4$

c) $5x - 8 = 12$, $\frac{5x}{6} - \frac{4}{3} = 2$, and $\frac{5x}{6} - \frac{8}{6} = \frac{12}{6}$

5. Solve each equation. Verify each solution.

K **a)** $5x + 24 = 2x$ **c)** $-4x - 1 = -3x + 5$ **e)** $3b - 4 - 5b = -3b - 2$

b) $2k = 4k - 15$ **d)** $2x - 3x + 6 = 7 - x + 2$ **f)** $a + 2a + 3a - 6 = 7a - 6$

6. Solve $n + (n + 1) + (n + 2) = 54$.

7. Solve each equation. Verify each solution.

a) $3(x - 5) = 6$ **c)** $-3(5 - 6m) = 39$ **e)** $3(c + 5) = 4(1 - 2c)$

b) $-5 = 5(3 + 2d)$ **d)** $2(x - 2) = 3x - 14$ **f)** $4(x - 2) = -3(2x + 6)$

8. A number, n, decreased by 5, is equal to 3 times the number plus 1. Determine the number.

9. The perimeter of a rectangle is 36 cm. The width is 5 cm less than the length. Determine the dimensions of the rectangle.

10. George is three times as old as Sam. Five years from now, the sum of their ages will be 46.

a) Create an equation that represents the relationship between George's and Sam's ages five years from now.

b) Use your equation to determine their current ages.

11. Fill in the missing column for each equation.

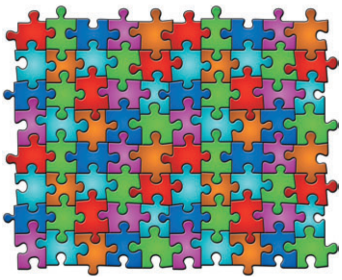
	Equation	Common Denominator of All Terms	Equation with Denominators Eliminated
a)	$\frac{3x}{4} + \frac{2}{3} = 2$		
b)	$\frac{1}{2} - \frac{x}{3} = \frac{1}{3}$		
c)	$\frac{2}{3} = 5 + x$		
d)	$\frac{x - 5}{4} + 1 = \frac{1}{2}$		
e)	$-16 = \frac{x}{5} + \frac{x}{3}$		
f)	$\frac{-2}{5}(x - 8) = 4$		
g)	$\frac{y + 2}{3} = \frac{1}{5}(2y + 3)$		

12. Solve each equation. Verify each solution.

a) $\frac{x}{3} = 2$ **c)** $\frac{x}{2} + \frac{x}{3} = 10$ **e)** $\frac{3k}{5} - 6 = \frac{k}{3}$

b) $\frac{d}{4} + 3 = 2$ **d)** $\frac{c}{3} - \frac{c}{4} = 3$ **f)** $\frac{2x + 1}{3} = 5$

13. The sum of one-half of a number, q, and three-fifths is two-thirds the number q. Determine the number.



14. For each of the following, create and solve an equation.

A **a)** It takes Eli 4 hours to paint a room. It takes Mia 3 hours to paint a room. How long would it take them to paint the room together?

b) Amir can put together a puzzle in 30 minutes. Bob takes double the amount of time. How long will it take them to do it together?

c) A jet left Toronto for Vancouver, travelling at a speed of 600 km/h. At the same time, a jet left Vancouver for Toronto, travelling at a speed of 800 km/h. If the distance between Toronto and Vancouver is 3500 km, when will the jets pass each other?

15. A square has sides of length $2k - 1$ units. An equilateral triangle has sides of length $k + 2$ units. The square and the triangle have the same perimeter. What is the value of k?

16. Show that the equation $2x - 3 = 4 + 2x$ has no solution. Why do
T you think this happens?

17. Show that the equation $\frac{10 - 6x}{2} = 5 - 3x$ has an infinite number of solutions. Why do you think this happens?

18. Jennifer solved the equation $\frac{4x - 1}{4} + \frac{2x - 1}{5} = 2$ below. Explain
C the mathematical operation she used in each step.

$$
\begin{aligned}
\frac{4x - 1}{4} + \frac{2x - 1}{5} &= 2 \\
20\left(\frac{4x - 1}{4}\right) + 20\left(\frac{2x - 1}{5}\right) &= 20(2) && \text{Step A} \\
5(4x - 1) + 4(2x - 1) &= 40 && \text{Step B} \\
20x - 5 + 8x - 4 &= 40 && \text{Step C} \\
28x - 9 &= 40 && \text{Step D} \\
28x - 9 + 9 &= 40 + 9 && \text{Step E} \\
28x &= 49 && \text{Step F} \\
\frac{28x}{28} &= \frac{49}{28} && \text{Step G} \\
x &= \frac{49}{28} \\
x &= \frac{7}{4} \text{ or } 1\frac{3}{4} && \text{Step H}
\end{aligned}
$$

19. Samir thinks that solving an equation with x on both sides is like solving an equation where x is only on one side. Do you agree or disagree with Samir? Use an example to justify your answer.

Extending

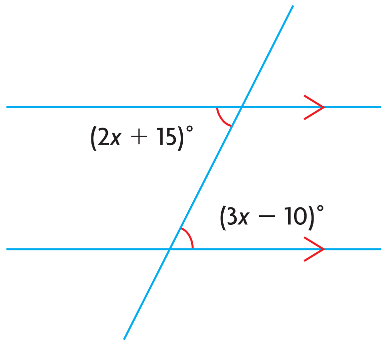
20. David has 16 dimes and quarters. Colin has twice as many dimes and $\frac{1}{3}$ as many quarters as David.

They both have the same amount of money. What coins does each boy have?

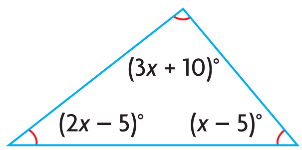
	Number of Quarters	Number of Dimes	Value of Quarters (¢)	Value of Dimes (¢)
David	q	$16 - q$	$25q$	
Colin	$\frac{q}{3}$			

21. Determine the value of x in each diagram.

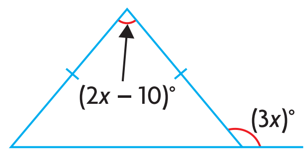
a)



b)



c)



22. Solve the equations:

a) $3x^2 - 2 = 25$

b) $2(x + 1)^2 - 1 = 71$

23. Chiaki is organizing a candy hunt for the children in her neighbourhood. She spent \$102 to buy 500 large candies and 400 small candies. The ratio of the price of a large candy to the price of a small candy is 7:4. Find the prices of one large and one small candy.