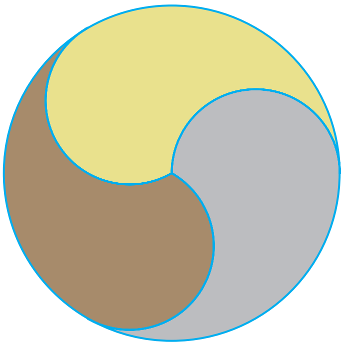
2.1 Midpoint of a Line Segment

YOU WILL NEED

- grid paper, ruler, and compass, or dynamic geometry software

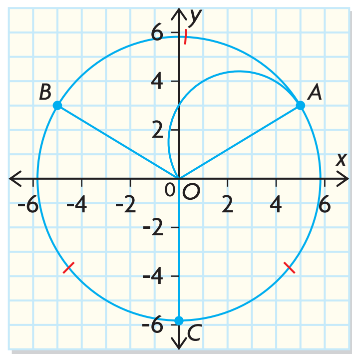


GOAL

Develop and use the formula for the midpoint of a line segment.

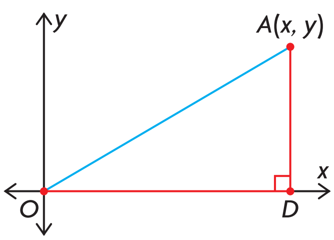
INVESTIGATE the Math

Ken's circular patio design for a client is shown at the left. He is planning the layout on a grid. He starts by drawing a circle that is centred at the origin. Then he marks points A, B, and C on the **circumference** of the circle to divide it into thirds. He joins these points to point O, at the centre of the circle. He needs to draw semicircles on the three **radii**: OA, OB, and OC.



? How can Ken determine the coordinates of the centre of the semicircle he needs to draw on radius OA?

A. Construct a line segment like OA on a coordinate grid, with O at (0, 0) and A at a grid point. Name the coordinates of $A(x, y)$.



B. Draw right triangle OAD, with side OD on the x-axis and side OA as the hypotenuse.

C. Draw a vertical line from E, the **midpoint** of OD, to M, the midpoint of OA. Explain why $\triangle OME$ is similar to $\triangle OAD$. Explain how the sides of the triangles are related. Estimate the coordinates of M.

D. Record the coordinates of point M. Explain why this is the centre of the semicircle that Ken needs to draw.

Reflecting

E. Why does it make sense that the coordinates of point M are the means of the coordinates of points O and A?

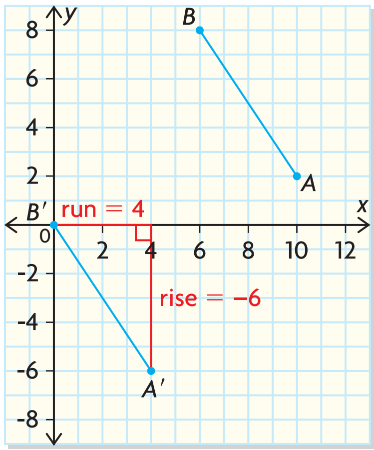
F. Suppose that point O had not been at (0, 0) but at another point instead. If (x_1, y_1) and (x_2, y_2) are endpoints of a line segment, what formula can you write to represent the coordinates of the midpoint? Why does your formula make sense?

APPLY the Math

EXAMPLE 1 **Reasoning** about the midpoint formula when one endpoint is not the origin

Determine the midpoint of a line segment with endpoints $A(10, 2)$ and $B(6, 8)$.

Robin's Solution: Using translations



$B'(6 - 6, 8 - 8) = B'(0, 0)$
$A'(10 - 6, 2 - 8) = A'(4, -6)$

I drew *AB* by plotting points *A* and *B* on a grid and joining them.

To make it easier to calculate the midpoint of *AB*, I decided to translate *AB* so that one endpoint would be at the origin. I moved point *B* to the origin by translating it 6 units left and 8 units down. I did the same to point *A* to get (4, −6) for *A*′.

I could see that the run of *A*′*B*′ was 4 and the rise was −6.

x-coordinate of midpoint M'

$= 0 + \frac{4}{2}$

$= 2$

I determined the *x*-coordinate of the midpoint of *A*′*B*′ by adding half the run to the *x*-coordinate of *B*′.

y-coordinate of midpoint M'

$= 0 + \frac{-6}{2}$

$= -3$

I determined the *y*-coordinate of the midpoint of *A*′*B*′ by adding half the rise to the *y*-coordinate of *B*′.

The midpoint of line segment $A'B'$ is $(2, -3)$.

$M_{AB} = M(2 + 6, -3 + 8)$
$M_{AB} = (8, 5)$

To determine the coordinates of *M*, the midpoint of *AB*, I had to undo my translation. I added 6 to the *x*-coordinate of the midpoint and 8 to the *y*-coordinate.

The midpoint of line segment AB is $(8, 5)$.

Sarah's Solution: Calculating using a formula

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

I decided to use the midpoint formula.

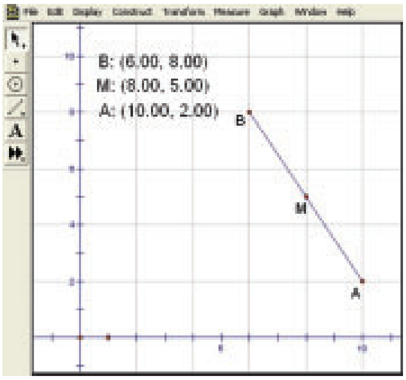
$$x_1 = 10, y_1 = 2$$
$$x_2 = 6, y_2 = 8$$

I chose point $A(10, 2)$ to be (x_1, y_1) and point $B(6, 8)$ to be (x_2, y_2).

$$\begin{aligned}(x, y) &= \left(\frac{10 + 6}{2}, \frac{2 + 8}{2}\right)\\ &= \left(\frac{16}{2}, \frac{10}{2}\right)\\ &= (8, 5)\end{aligned}$$

I substituted these values into the midpoint formula.

The midpoint of line segment AB is (8, 5).



I verified my calculations by constructing AB using dynamic geometry software. Then I constructed the midpoint and measured the coordinates of all three points. My calculations were correct.

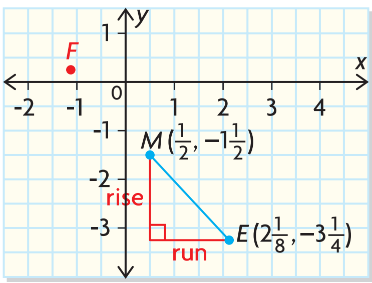
Tech | Support

For help constructing and labelling a line segment, displaying coordinates, and constructing the midpoint using dynamic geometry software, see Appendix B-21, B-22, B-20, and B-30.

EXAMPLE 2 Reasoning to determine an endpoint

Line segment EF has an endpoint at $E\left(2\frac{1}{8}, -3\frac{1}{4}\right)$. Its midpoint is located at $M\left(\frac{1}{2}, -1\frac{1}{2}\right)$. Determine the coordinates of endpoint F.

Ali's Solution

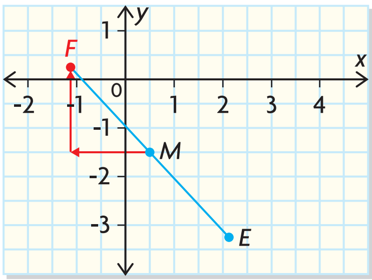


I reasoned that if I could calculate the run and rise between E and M, adding these values to the x- and y-coordinates of M would give me the x- and y-coordinates of F.

$$\begin{aligned}\text{Run} &= x_2 - x_1\\ &= 2\frac{1}{8} - \frac{1}{2}\\ &= \frac{17}{8} - \frac{4}{8}\\ &= \frac{13}{8} \text{ or } 1\frac{5}{8}\end{aligned}$$

$$\begin{aligned}\text{Rise} &= y_2 - y_1\\ &= -3\frac{1}{4} - \left(-1\frac{1}{2}\right)\\ &= -\frac{13}{4} + \frac{6}{4}\\ &= -\frac{7}{4} \text{ or } -1\frac{3}{4}\end{aligned}$$

I let $M = (x_1, y_1)$ and $E = (x_2, y_2)$. Then I calculated the rise and the run.



To get to F, I had to start at M and move $1\frac{5}{8}$ units left and $1\frac{3}{4}$ units up.

x-coordinate of F

$$\begin{aligned}&= \frac{1}{2} - 1\frac{5}{8}\\ &= \frac{4}{8} - \frac{13}{8}\\ &= -\frac{9}{8} \text{ or } -1\frac{1}{8}\end{aligned}$$

y-coordinate of F

$$\begin{aligned}&= -1\frac{1}{2} + 1\frac{3}{4}\\ &= -\frac{6}{4} + \frac{7}{4}\\ &= \frac{1}{4}\end{aligned}$$

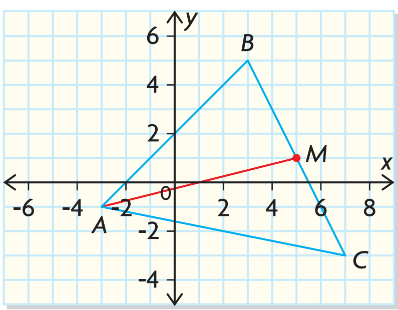
I subtracted $1\frac{5}{8}$ from the x-coordinate of M and added $1\frac{3}{4}$ to the y-coordinate.

The coordinates of F are $\left(-1\frac{1}{8}, \frac{1}{4}\right)$.

EXAMPLE 3 **Connecting** the midpoint to an equation of a line

A triangle has vertices at $A(-3, -1)$, $B(3, 5)$, and $C(7, -3)$. Determine an equation for the **median** from vertex A.

Graeme's Solution



I plotted A, B, and C and joined them to create a triangle.

I saw that the side opposite vertex A is BC, so I estimated the location of the midpoint of BC. I called this point M. Then I drew the median from vertex A by drawing a straight line from point A to M.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$M_{BC} = \left(\frac{3 + 7}{2}, \frac{5 + (-3)}{2}\right)$$

$$= (5, 1)$$

I used the midpoint formula to calculate the coordinates of *M*.

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{AM} = \frac{1 - (-1)}{5 - (-3)}$$

$$= \frac{2}{8}$$

$$= \frac{1}{4}$$

To determine the equation of *AM*, I had to calculate its slope. I used the coordinates of *A* as (x_1, y_1) and the coordinates of *M* as (x_2, y_2) in the slope formula.

An equation for *AM* is

$$y = \frac{1}{4}x + b$$

I substituted the slope of *AM* for *m* in $y = mx + b$.

$$-1 = \frac{1}{4}(-3) + b$$

Then I determined the value of *b* by substituting the coordinates of *A* into the equation and solving for *b*.

$$-1 + \frac{3}{4} = b$$

$$-\frac{1}{4} = b$$

The equation of the median is $y = \frac{1}{4}x - \frac{1}{4}$.

EXAMPLE 4 Solving a problem using midpoints

A waste management company is planning to build a landfill in a rural area. To balance the impact on the two closest towns, the company wants the landfill to be the same distance from each town. On a coordinate map of the area, the towns are at $A(1, 8)$ and $B(5, 2)$. Describe all the possible locations for the landfill.

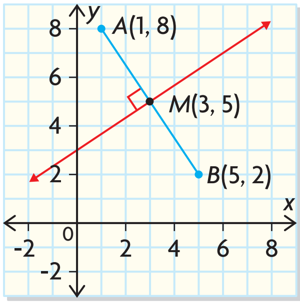
Wendy's Solution

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$M_{AB} = \left(\frac{1 + 5}{2}, \frac{8 + 2}{2}\right)$$

$$= (3, 5)$$

I used the midpoint formula to determine the coordinates of the midpoint of *AB*.



I drew AB on a grid. I knew that the points equally far from A and B lie on the **perpendicular bisector** of AB, so I added this to my sketch.

$$(x_1, y_1) = A(1, 8)$$
$$(x_2, y_2) = B(5, 2)$$

$$\begin{aligned}\text{Slope of } AB &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{2 - 8}{5 - 1}\\ &= \frac{-6}{4}\\ &= -\frac{3}{2}\end{aligned}$$

I needed the slope of the perpendicular bisector so that I could write an equation for it. I used the slope formula to determine the slope of AB.

The slope of the perpendicular bisector is $\frac{2}{3}$.

Since the perpendicular bisector is perpendicular to AB, its slope is the negative reciprocal of the slope of AB.

An equation for the perpendicular bisector is

$$\begin{aligned}y &= \frac{2}{3}x + b\\ 5 &= \frac{2}{3}(3) + b\\ 5 &= 2 + b\\ 5 - 2 &= b\\ 3 &= b\end{aligned}$$

To determine the value of b, I substituted the coordinates of the midpoint of AB into the equation and solved for b. This worked because the midpoint is on the perpendicular bisector, even though points A and B aren't.

Therefore, $y = \frac{2}{3}x + 3$ is the equation of the perpendicular bisector. Possible locations for the landfill are determined by points that lie on the line with equation $y = \frac{2}{3}x + 3$.

Communication *Tip*

A perpendicular bisector is also called a right bisector.

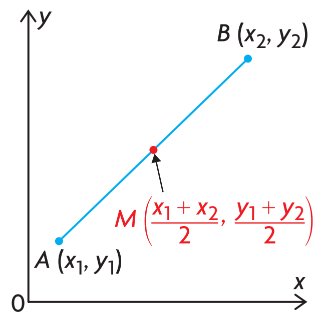
In Summary

Key Idea

- The coordinates of the midpoint of a line segment are the means of the coordinates of the endpoints.

Need to Know

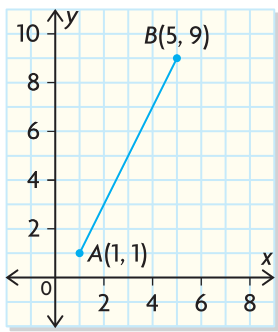
- The formula $(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ can be used to calculate the coordinates of a midpoint.
- The coordinates of a midpoint can be used to determine an equation for a median in a triangle or the perpendicular bisector of a line segment.



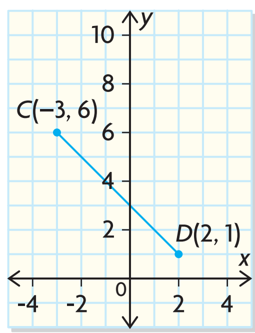
CHECK Your Understanding

1. Determine the coordinates of the midpoint of each line segment, using one endpoint and the rise and run. Verify the midpoint by measuring with a ruler.

a)

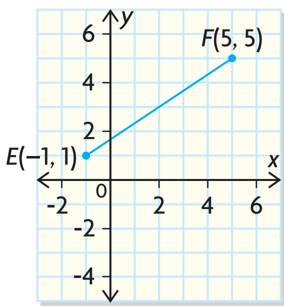


b)

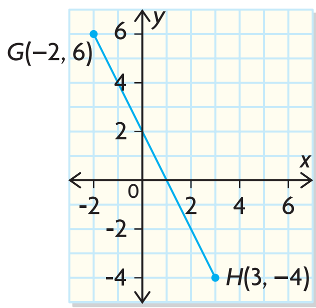


2. Determine the coordinates of the midpoint of each line segment.

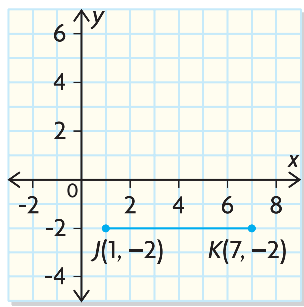
a)



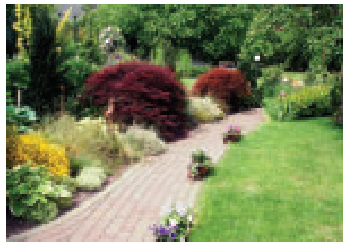
b)



c)



3. On the design plan for a landscaping project, a straight path runs from (11, 29) to (53, 9). A lamp is going to be placed halfway along the path.
   a) Draw a diagram that shows the path.
   b) Determine the coordinates of the lamp on your diagram.



PRACTISING

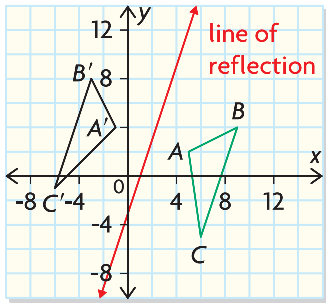
4. Determine the coordinates of the midpoint of the line segment with each pair of endpoints.
   a) $A(-1, 3)$ and $B(5, 7)$
   b) $J(-2, 3)$ and $K(3, 4)$
   c) $X(6, -2)$ and $Y(-2, -2)$
   d) $P(2, -4)$ and $I(-3, 5)$
   e) $U\left(\frac{1}{2}, -\frac{3}{2}\right)$ and $V\left(-\frac{5}{2}, -\frac{1}{2}\right)$
   f) $G(1.5, -2.5)$ and $H(-1, 4)$
5. The endpoints of the diameter of a circle are $A(-1, 1)$ and $B(2.5, -3)$. Determine the coordinates of the centre of the circle.
6. $P(-3, -1)$ is one endpoint of PQ. $M(1, 1)$ is the midpoint of PQ. Determine the coordinates of endpoint Q. Explain your solution.
7. A triangle has vertices at $A(2, -2)$, $B(-4, -4)$, and $C(0, 4)$.
   K a) Draw the triangle, and determine the coordinates of the midpoints of its sides.
   b) Draw the median from vertex A, and determine its equation.
8. A radius of a circle has endpoints $O(-1, 3)$ and $R(2, 2)$. Determine the endpoints of the diameter of this circle. Describe any assumptions you make.
9. A quadrilateral has vertices at $P(1, 3)$, $Q(6, 5)$, $R(8, 0)$, and $S(3, -2)$. Determine whether the diagonals have the same midpoint.
10. **C** Mayda is sketching her design for a rectangular garden. By mistake, she has erased the coordinates of one of the corners of the garden. As a result, she knows only the coordinates of three of the rectangle's vertices. Explain how Mayda can use midpoints to determine the unknown coordinates of the fourth vertex of the rectangle.
11. A triangle has vertices at $P(7, 7)$, $Q(-3, -5)$, and $R(5, -3)$.
    A a) Determine the coordinates of the midpoints of the three sides of $\triangle PQR$.
    b) Calculate the slopes of the **midsegments** of $\triangle PQR$.
    c) Calculate the slopes of the three sides of $\triangle PQR$.
    d) Compare your answers for parts b) and c). What do you notice?
12. Determine the equations of the medians of a triangle with vertices at $K(2, 5)$, $L(4, -1)$, and $M(-2, -5)$.

13. Determine an equation for the perpendicular bisector of a line segment with each pair of endpoints.
 a) $C(-2, 0)$ and $D(4, -4)$
 b) $A(4, 6)$ and $B(12, -4)$
 c) $L(-2, -4)$ and $M(8, 4)$
 d) $Q(-5, 6)$ and $R(1, -2)$

14. A committee is choosing a site for a county fair. The site needs to be located the same distance from the two main towns in the county. On a map, these towns have coordinates (3, 10) and (13, 4). Determine an equation for the line that shows all the possible sites for the fair.

15. A triangle has vertices at $D(8, 7)$, $E(-4, 1)$, and $F(8, 1)$. Determine the coordinates of the point of intersection of the medians.
T

16. In the diagram, $\triangle A'B'C'$ is a reflection of $\triangle ABC$. The coordinates of all vertices are integers.
 a) Determine the equation of the line of reflection.
 b) Determine the equations of the perpendicular bisectors of AA', BB', and CC'.
 c) Compare your answers for parts a) and b). What do you notice?



Health *Connection*

Vegetables, a source of vitamins and minerals, lower blood pressure, reduce the risk of stroke and heart disease, and decrease the chance of certain types of cancer.

17. A quadrilateral has vertices at $W(-7, -4)$, $X(-3, 1)$, $Y(4, 2)$, and $Z(-2, -7)$. Two lines are drawn to join the midpoints of the non-adjacent sides in the quadrilateral. Determine the coordinates of the point of intersection of these lines.

18. Describe two different strategies you can use to determine the coordinates of the midpoint of a line segment using its endpoints. Explain how these strategies are similar and how they are different.

Extending

19. A point is one-third of the way from point $A(1, 7)$ to point $B(10, 4)$. Determine the coordinates of this point. Explain the strategy you used.

20. A triangle has vertices at $S(6, 6)$, $T(-6, 12)$, and $U(0, -12)$. SM is the median from vertex S.
 a) Determine the coordinates of the point that is two-thirds of the way from S to M that lies on SM.
 b) Repeat part a) for the other two medians, TN and UR.
 c) Show that the three medians intersect at a common point. What do you notice about this point?
 d) Do you think the relationship you noticed is true for all triangles? Explain.