3.3 Factored Form of a Quadratic Relation

YOU WILL NEED

- grid paper and ruler, or graphing calculator

Career *Connection*

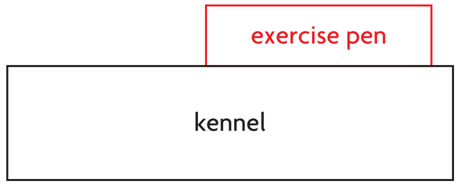
Jobs at a dog kennel include kennel technician, veterinary technician, consultant, groomer, dog walker, and secretary.

GOAL

Relate the factors of a quadratic relation to the key features of its graph.

INVESTIGATE the Math

Boris runs a dog kennel. He has purchased 80 m of fencing to build an outdoor exercise pen against the wall of the kennel.



? What dimensions should Boris use to maximize the area of the exercise pen?

A. If x represents the width of the pen, write an expression for its length.

B. Write a relation, in terms of x, for the area of the exercise pen. Identify the factors of the relation.

C. Create a table of values, and graph the relation you wrote for part B.

D. Use your table of values or graph to verify that the area relation is quadratic.

E. Does the relation have a maximum value or a minimum value? Explain how you know.

F. Determine the zeros of the parabola.

G. Determine the equation of the axis of symmetry of the parabola.

H. Determine the vertex of the parabola.

I. What are the dimensions that maximize the area of the exercise pen?

Reflecting

J. How are the factors of this relation related to the zeros of the graph?

K. The area relation can also be written as $A = -2(x)(x - 40)$ or $A = -2(x - 0)(x - 40)$, by dividing out the common factor of -2 from one of the factors. Explain why the **factored form of a quadratic relation** is useful when graphing the relation by hand.

factored form of a quadratic relation

a quadratic relation that is written in the form $y = a(x - r)(x - s)$

L. What is the area of the largest exercise pen that Boris can build?

APPLY the Math

EXAMPLE 1 **Reasoning** about the nature of a relation

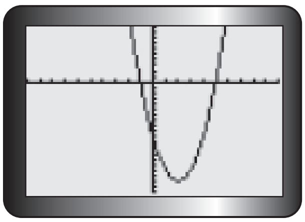
Is the graph of $y = 2(x + 1)(x - 5)$ a parabola? If so, in what direction does it open? Justify your answer.

Jasper's Solution

x	y	First Difference	Second Difference
−3	32		
		$14 - 32 = -18$	
−2	14		$-14 - (-18) = 4$
		−14	
−1	0		4
		−10	
0	−10		4
		−6	
1	−16		4
		−2	
2	−18		4
		2	
3	−16		

I created a table of values. Then I calculated the first and second differences. The second differences are constant but not zero, and they are also positive.

I predict that the graph of this relation is a parabola that opens upward.



I used a graphing calculator to graph the relation and check my predictions.

My predictions were correct.

EXAMPLE 2 **Selecting a strategy** to graph a quadratic relation given in factored form

Determine the y-intercept, zeros, axis of symmetry, and vertex of the quadratic relation $y = 2(x - 4)(x + 2)$. Then sketch the graph.

Cindy's Solution

$$y = 2(x - 4)(x + 2)$$
$$y = 2(0 - 4)(0 + 2)$$
$$y = 2(-4)(2)$$
$$y = -16$$

To determine the y-intercept, I substituted $x = 0$ into the equation. I noticed that multiplying the numbers in the original equation would have given the same result.

The y-intercept occurs at $(0, -16)$.

$0 = 2(x - 4)(x + 2)$

$x - 4 = 0$ or $x + 2 = 0$

$x = 4 \qquad x = -2$

The zeros occur at $(4, 0)$ and $(-2, 0)$.

To determine the zeros, I let $y = 0$. I know that a product is zero only when one of its factors is zero, so I set each factor equal to 0 and solved for x.

$$x = \frac{4 + (-2)}{2}$$

$x = 1$

The equation of the axis of symmetry is $x = 1$.

The axis of symmetry passes through the midpoint of the zeros, so I calculated the mean.

$y = 2(x - 4)(x + 2)$

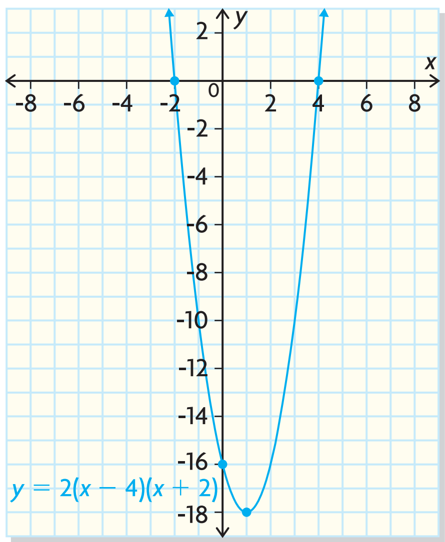
$y = 2(1 - 4)(1 + 2)$

$y = 2(-3)(3)$

$y = -18$

The vertex is $(1, -18)$.

The vertex lies on the axis of symmetry, so its x-coordinate is 1. I substituted $x = 1$ into the equation of the parabola to determine the y-coordinate.



I plotted the y-intercept, zeros, and vertex. Then I joined the points with a smooth curve.

EXAMPLE 3 Selecting a strategy to graph a quadratic relation given in factored form

Determine the y-intercept, zeros, axis of symmetry, and vertex of the quadratic relation $y = (x - 2)^2$. Then sketch the graph.

Kylie's Solution

$y = (x - 2)^2$

$y = (0 - 2)^2$

$y = 4$

The y-intercept occurs at $(0, 4)$.

To determine the y-intercept, I substituted $x = 0$ into the equation and solved for y.

$y = (x - 2)^2$
$0 = (x - 2)^2$
$0 = x - 2$
$x = 2$
The zero occurs at (2, 0).

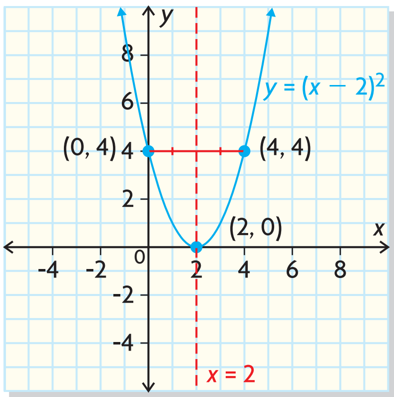
To determine the zeros, I let $y = 0$ and solved for x. Both factors are the same, since $y = (x - 2)^2$ is the same as $y = (x - 2)(x - 2)$. There is only one solution to $0 = x - 2$, so there is only one zero for the quadratic relation.

The equation of the axis of symmetry is $x = 2$.

The axis of symmetry passes through the midpoint of the zeros. Since there is only one zero, the axis of symmetry must pass through it.

The vertex is (2, 0).

Since (2, 0) is on the line $x = 2$, this point is also the vertex.

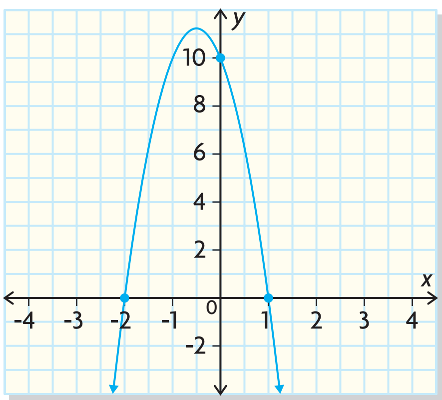


The y-intercept, (0, 4), is 2 units to the left of the axis of symmetry. There must be another point with y-coordinate 4 on the parabola, 2 units to the right of $x = 2$. This point is (4, 4).

I plotted these three points and joined them with a smooth curve.

EXAMPLE 4 Connecting the features of a parabola to its equation

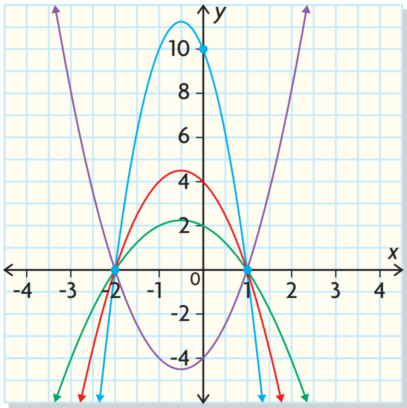
Determine an equation for this parabola.



Petra's Solution

The zeros occur at $(-2, 0)$ and $(1, 0)$.
$y = a(x - r)(x - s)$
$y = a[x - (-2)](x - 1)$
$y = a(x + 2)(x - 1)$

I determined the zeros of the parabola and substituted them into the factored form of a quadratic relation. I did this because I know that a parabola is described by a quadratic relation.



There are infinitely many parabolas with these zeros, all with different *y*-intercepts. A few examples are shown in the diagram. To determine the equation of the given parabola, I need to determine the value of *a*. This is the only value that varies in my equation $y = a(x + 2)(x - 1)$.

y-intercept occurs at (0, 10).

$$\begin{aligned} y &= a(x + 2)(x - 1) \\ 10 &= a(0 + 2)(0 - 1) \\ 10 &= a(2)(-1) \\ 10 &= -2a \\ -5 &= a \end{aligned}$$

I chose the *y*-intercept to substitute into my equation because its coordinates are integers. Then I solved for *a*.

An equation for the given parabola is $y = -5(x + 2)(x - 1)$.

I substituted the value of *a* into my equation.

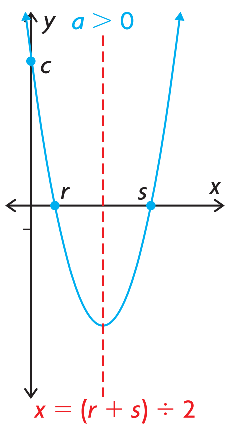
In Summary

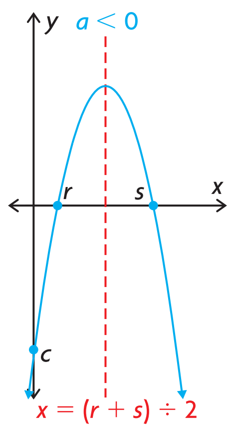
Key Ideas

- When a quadratic relation is expressed in factored form $y = a(x - r)(x - s)$, each factor can be used to determine a zero, or *x*-intercept, of the parabola.
- An equation for a parabola can be determined using the zeros and the coordinates of one other point on the parabola.

Need to Know

- If a quadratic relation is expressed in the form $y = a(x - r)(x - s)$,
  - the *x*-intercepts are *r* and *s*
  - the equation of the axis of symmetry is the vertical line defined by the equation $x = (r + s) \div 2$
  - the *x*-coordinate of the vertex is $(r + s) \div 2$
  - the *y*-intercept is $c = a \times r \times s$





CHECK Your Understanding

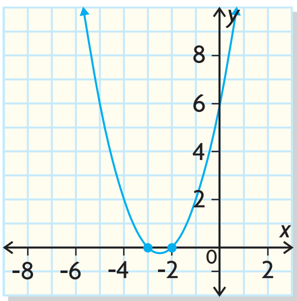
1. Complete the following for each quadratic relation below.

i) Determine the zeros.
ii) Explain how the zeros are related to the factors in the quadratic expression.
iii) Determine the y-intercept.
iv) Determine the equation of the axis of symmetry.
v) Determine the coordinates of the vertex.
vi) Is the graph a parabola? How can you tell?
vii) Sketch the graph.

a) $y = -2x(x + 3)$
b) $y = (x - 3)(x + 1)$
c) $y = 2(x - 1)(x + 2)$

2. Match each quadratic relation with the correct parabola.

a) $y = (x - 2)(x + 3)$
b) $y = (x - 3)(x + 2)$
c) $y = (x + 2)(x + 3)$
d) $y = (3 - x)(x + 2)$
e) $y = (3 + x)(2 - x)$
f) $y = (x - 2)(x - 3)$

i)


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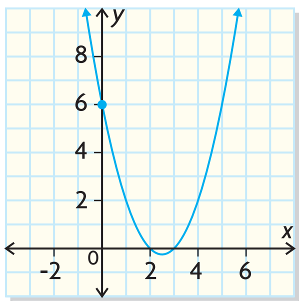

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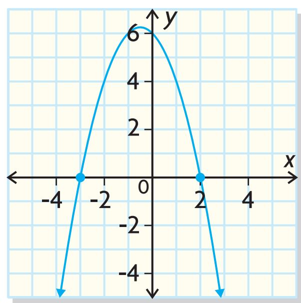

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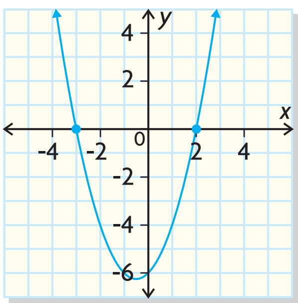

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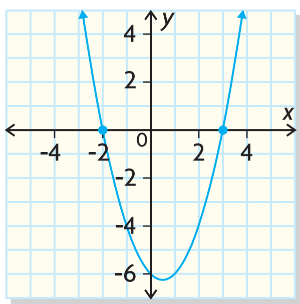

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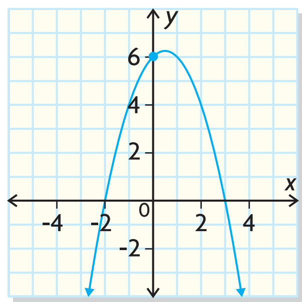


3. A quadratic relation has an equation of the form $y = a(x - r)(x - s)$. The graph of the relation has zeros at $(2, 0)$ and $(-6, 0)$ and passes through the point $(3, 5)$. Determine the value of a.

PRACTISING

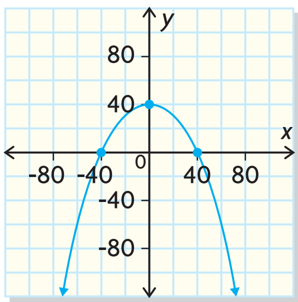
4. Determine the y-intercept, zeros, equation of the axis of symmetry, and vertex of each quadratic relation.
   - a) $y = (x - 3)(x + 3)$
   - b) $y = (x + 2)(x + 2)$
   - c) $y = (x - 2)(x - 2)$
   - d) $y = -(x - 2)(x + 2)$
   - e) $y = 2(x + 3)^2$
   - f) $y = -4(x - 4)^2$

5. Sketch the graph of each relation in question 4.

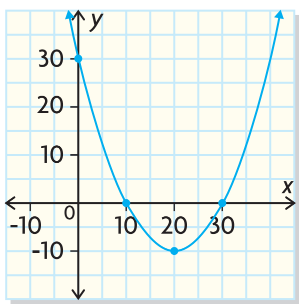
6. A quadratic relation has an equation of the form $y = a(x - r)(x - s)$. Determine the value of a when
   - a) the parabola has zeros at $(4, 0)$ and $(2, 0)$ and a y-intercept at $(0, 1)$
   - b) the parabola has x-intercepts at $(4, 0)$ and $(-2, 0)$ and a y-intercept at $(0, -1)$
   - c) the parabola has zeros at $(5, 0)$ and $(0, 0)$ and a minimum value of -10
   - d) the parabola has x-intercepts at $(5, 0)$ and $(-3, 0)$ and a maximum value of 6
   - e) the parabola has its vertex at $(5, 0)$ and a y-intercept at $(0, -10)$

7. **K** Determine the zeros, equation of the axis of symmetry, and vertex of each parabola. Then determine an equation for each quadratic relation.

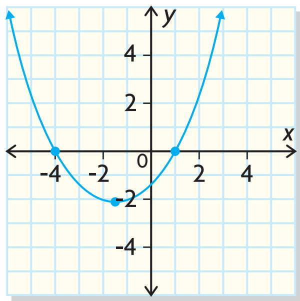
a)



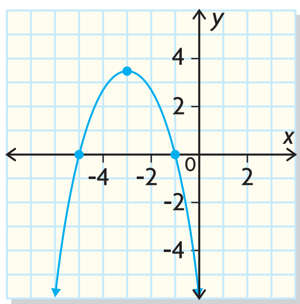
b)



c)



d)



8. **a)** Sketch the graph of $y = a(x - 2)(x + 3)$ when $a = 3$.
b) Describe how your graph for part a) would change if the value of a changed to 2, 1, 0, -1, -2, and -3.

9. **a)** Sketch the graph of $y = (x - 2)(x - s)$ when $s = 3$.
b) Describe how your graph for part a) would change if the value of s changed to 2, 1, 0, 1, -2, and -3.

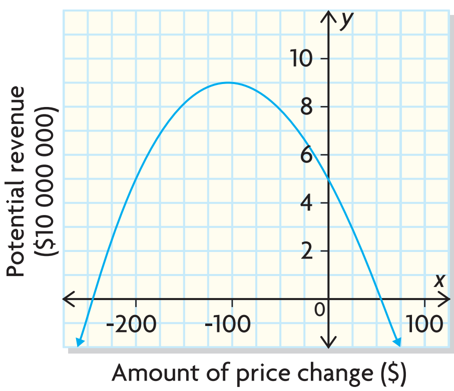
10. The x-intercepts of a parabola are -3 and 5. The parabola crosses the y-axis at -75.
a) Determine an equation for the parabola.
b) Determine the coordinates of the vertex.

11. Sometimes the equation $y = a(x - r)(x - s)$ cannot be used to determine the equation of a parabola from its graph. Explain when this is not possible, and draw graphs to illustrate.

12. A ball is thrown into the air from the roof of a building that is 25 m high. The ball reaches a maximum height of 45 m above the ground after 2 s and hits the ground 5 s after being thrown.

a) Use the fact that the relation between time and the height of the ball is a quadratic relation to sketch an accurate graph of the relation.
b) Carefully fold the graph along its axis of symmetry. Extend the short side of the parabola to match the long side.
c) Where does the extended graph cross the time axis?
d) What are the zeros of the relation?
e) Determine the coordinates of the vertex.
f) Determine an equation for the relation.
g) What is the meaning of each zero?

13. A car manufacturer decides to change the price of its new luxury sedan (LS) model to increase sales. The graph shows the relationship between revenue and the size of the price change.



a) Determine an equation for the graph.
b) How should the price be changed for maximum revenue?

Career Connection

Jobs in the music industry include recording artist, studio musician, songwriter, producer, recording engineer, digital audio workstation operator, music programmer, and re-mixer.

14. **T** Ryan owns a small music store. He currently charges \$10 for each CD. At this price, he sells about 80 CDs a week. Experience has taught him that a \$1 increase in the price of a CD means a drop of about five CDs per week in sales. At what price should Ryan sell his CDs to maximize his revenue?

15. **A** Rahj owns a hardware store. For every increase of 10¢ in the price of a package of batteries, he estimates that sales decrease by 10 packages per day. The store normally sells 700 packages of batteries per day, at \$5.00 per package.

a) Determine an equation for the revenue, y, when x packages of batteries are sold.

b) What is the maximum daily revenue that Rahj can expect from battery sales?

c) How many packages of batteries are sold when the revenue is at a maximum?

16. **C** Create a flow chart that summarizes the process you would use to determine an equation of a parabola from its graph. Assume that the parabola has two zeros.

Extending

17. Without graphing, match each quadratic relation in factored form (column 1) with the equivalent quadratic relation in standard form (column 2). Explain your reasoning.

Column 1	Column 2
a) $y = (2x - 3)(x + 4)$	**i)** $y = 12x^2 - 5x - 3$
b) $y = (3x + 1)(4x - 3)$	**ii)** $y = -2x^2 - 5x + 12$
c) $y = (3 - 2x)(4 + x)$	**iii)** $y = 2x^2 + 11x - 12$
d) $y = (3 - 4x)(1 + 3x)$	**iv)** $y = 2x^2 + 5x - 12$
	v) $y = -12x^2 + 5x + 3$
	vi) $y = 12x^2 + 5x - 3$

18. Martin wants to enclose the backyard of his house on three sides to form a rectangular play area. He is going to use the wall of his house and three sections of fencing. The fencing costs \$15/m, and Martin has budgeted \$720. Determine the dimensions that will produce the largest rectangular area.