SL2 Lesson 17 – Optimization

Calculus - Santowski

Lesson Objectives

- 1. Solve optimization problems using a variety of calculus and non-calculus based strategies
- 2. Solve optimization problems with and without the use of the TI 84 GDC
- 3. Work with problems set in mathematical contexts or in the context of a real world problem

Fast Five

- 1. Find the maximum point of $M = x(30-2x)^2$ on [0,30]
-
- 2. If $S(r,h) = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h = 1000$, find an expression for S(r)
- 3. An 8 cm piece of wire is to be cut into 2 pieces, one to be formed into a circle and one to be formed into a square. Find an expression for the total area enclosed by the two figures

(A) Review

- When a function has a local max/min at $x = c$, then the value of the derivative is 0 at c
- However, the converse ➔ when a function has a zero first derivative, the critical point may or may not be an extreme value (i.e. max/min)
- So we need to "test" whether the critical point is a max or a min or neither ➔ thus we have either our first derivative test or our second derivative test
- In the first derivative test, we look for sign changes in the derivative values before and after the critical point ➔ if the derivative changes from +ve to –ve, then the fcn has a max and vice versa for a min
- In the second derivative test, we test for concavity ➔ a fcn will have a max only if the curve is concave down, so if the second derivative is negative, then the critical point is a maximum and vice versa for a min

Intro to Our Examples

- We will work through several examples together, wherein we outline the key steps involved in the process of determining max/min values in application based problems

(B) Optimizing Area - Example

- We will work through the following problem together.
- **ex 1. A farmer wishes to enclose a rectangular lot with area being 600 m^2. What is the least amount of fencing that will be needed?**
- First, before we start with any "math", we need to visualize & understand the problem.

(B) Optimizing Area - Example

- **ex 1. A farmer wishes to enclose a rectangular lot with area being 600 m². What is the least amount of fencing that will be needed?**

- (a) Show that p(L) = 2L + 1200/L
- (b) Find value(s) for L that will minimize the amount of fencing required.
- (c) Verify that your value(s) for L do in fact correspond to a minimal value (use either FDT or SDT)

(B) Optimizing Area - Example

- **ex 1. A farmer wishes to enclose a rectangular lot each being 600 m². What is the least amount of fencing that will be needed?**

- Start with key relationships ➔ p = 2(L + W) and A = L×W
- The reason for looking for a second relationships is that my first relationship (perimeter) has two variables, so I would want to make a substitution giving me a relationship with only one variable
- Since I wish to optimize the perimeter ➔ p = 2(L + W)
- Now I use the relationship A = 600 = L×W ➔ so W = 600/L

- Then p(L) = 2L + 2(600/L) = 2L + $1200L^{-1}$

(B) Optimizing Area - Example

- **ex 1. A farmer wishes to enclose a rectangular lot each being 600 m^2. What is the least amount of fencing that will be needed?**

- Given that $p(L) = 2L + 2(600/L) = 2L + 1200L^{-1}$
- Since we want to "optimize" the fencing (use the least amount), we want to find the minimum point on the graph of $y = p(L)$
- So we will differentiate and set the derivative to 0

- $d/dL\,(p(L)) = d/dL\,(2L + 1200L^{-1}) = 2 - 1200L^{-2}$

- So $p`(L) = 0 = 2 - 1200L^{-2}$ ➔ thus $L^2 = 600$ or $L = \sqrt{600} = 24.5$ m

(B) Optimizing Area - Example

- Now, we need to verify that the critical point that we generated (when L = 24.5m) is actually a minimum point.
- We can use the first derivative test and we can substitute L = 20m and L = 25m into the derivative equation
- p`(20) = 2 – 1200(20)-2 = 2 – 3 = -1
- p`(25) = 2 – 1200(25) -2 = 2 – 1.92 = +0.08
- So our derivative changes from –ve to +ve as L increases, so our function must have a minimum point at 24.5 m

- Alternatively, we can use the second derivative test
- p``(L) = -1200 ´ -2L-3 = 2400/L3
- Then p``(24.5) = 2400/(24.53) = +0.163 è which means that the second derivative is positive at L = 24.5m meaning our function is concave up which means that our critical value at L = 24.5 must correspond to a minimum

Strategy for solving max/min problems:

1. Understand the problem.
2. Develop a mathematical model of the problem
   a) sketch a diagram
   b) introduce variables to represent unknown quantities
   c) write an equation for the quantity to be maximized or minimized (optimized)
   d) write any additional equations that allow the dependent variables to be related; use it/them to reduce the number of independent unknowns in the optimizing function down to one variable; these relating equations often come from the problem statement itself, or the geometry of the diagram.
3. Determine the domain of the function.
4. Identify the critical points and the endpoints.
5. Use the SDT or the FDT to identify the candidate(s) as the x – coordinate of a maximum or minimum of the function to be optimized
6. Solve the mathematical model.
7. Interpret the solution.

(B) Optimizing Area – Questions

- ex 2. A farmer wishes to enclose three equal adjacent rectangular lots each being 600 m^2. What is the least amount of fencing that will be needed and what is the size of each lot?
- ex 3. A picture book will have pages of area 672 cm^2, with top and side margins of 3 cm and bottom margin of 4 cm. What should be the proportions of the page so that the greatest possible area is available for the pictures.
- ex 4. Sai-An is designing a poster for Mr. S's Calculus class that must have a 20 cm border at the top and bottom and a 15 cm border on the sides. The poster is to cover an area of 2.43 m^2. What are the dimensions of the banner that will maximize the non-border region of the poster?

(C) Optimizing Volume

- ex 5. If 2700 cm^2 of material is available to make an open topped box with a square base, find the largest possible volume of the box.
- First, before we start with any "math", we need to visualize & understand the problem.
- So once again, decide on what needs to be done
- Outline the steps of your strategy

(B) Optimizing Area - Example

- **ex 5. If 2700 cm^2 of material is available to make an open topped box with a square base, find the largest possible volume of the box?**
- (a) Show that $V(L) = 2700L – L^3$
- (b) Find value(s) for L that will maximize the volume of the box being made.
- (c) Verify that your value(s) for L do in fact correspond to a maximum value (use either FDT or SDT)

(C) Optimizing Volume

- ex 5. If 2700 cm^2 of material is available to make an open topped box with a square base, find the largest possible volume of the box.

- Start with what you want to "optimize" ➔ in this case the volume ➔ so we need a volume formula ➔ $V = L \times W \times H$
- We know that the base is a square (so $L = W$) ➔ so we can modify our formula as $V = L \times L \times H = L^2 \times H$
- But we are also given info. about the surface area (the material available to us), so we will work with a second relationship
- The surface area of a square based, on top box would be $SA = L\times L + 4L\times H$
- Thus $2700\ cm^2 = L^2 + LH$ then $H = (2700 - L^2)/L$

- So now we can finalize our formula, before differentiating as:
- $V(L) = L^2 \times (2700 - L^2)/L = L \times (2700 - L^2) = 2700L - L^3$
- But now let's consider the domain of our function ➔ $(0,\sqrt{2700})$

(D) Optimizing With Three Dimensional Geometric Shapes

- **Ex 6. A cylindrical can is made to hold one litre of oil. Find the dimensions of the can that will minimize the cost of the metal needed to manufacture the can.**

- First, before we start with any "math", we need to visualize & understand the problem.

- So once again, decide on what needs to be done
- Outline the steps of your strategy

(B) Optimizing Area - Example

- **Ex 6. A cylindrical can is made to hold one litre of oil. Find the dimensions of the can that will minimize the cost of the metal needed to manufacture the can.**

- (a) Show that $SA(r) = 2\pi r^2 + 2000/r$
- (b) Find value(s) for r that will minimize the surface area of the can being made.
- (c) Verify that your value(s) for r do in fact correspond to a maximum value (use either FDT or SDT)

(D) Optimizing With Three Dimensional Geometric Shapes

- Ex 6. A can is made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal needed to manufacture the can.

- we wish to minimize the surface area, so the surface area formula for cylinders is the place to start
- $SA(r,h) = 2\pi r^2 + 2\pi rh = 2\pi r(r + h)$
- Now again, we have 2 variables, so we need to express one in terms of the other
- The volume of a cylinder is $V = \pi r^2 h = 1000 \text{ cm}^3 = 1\text{L}$
- So $1000/\pi r^2 = h$
- Therefore $SA(r) = 2\pi r(r + 1000/\pi r^2) = 2\pi r^2 + 2000r^{-1}$
- And the domain would be? Well SA > 0, so $2\pi r^2 + 2000r^{-1} > 0$

(D) Optimizing With Three Dimensional Geometric Shapes

- Ex 6. A cylindrical can is made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal needed to manufacture the can.

- Now differentiate to find the critical points on the SA function
- $SA`(r) = 4\pi r - 2000r^{-2} = 0 \Rightarrow r^3 = 500/\pi \Rightarrow r = 5.42$ cm $\Rightarrow h = 10.8$ cm
- Now let's run through the first derivative test to verify a minimum SA
- $SA`(5) = 4\pi(5) - 2000(5)^{-2} = -17.2$
- $SA`(6) = 4\pi(6) - 2000(6)^{-2} = 19.8$
- So the first derivative changes from –ve to +ve, meaning that the value r = 5.42 cm does represent a minimum

(D) Optimizing With Three Dimensional Geometric Shapes

- Ex 6a. A cylindrical can is made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal needed to manufacture the can but the following conditions are included: the cost of the bottom is twice as much as the cost of the "sides" and the cost of the top is twice the cost of the bottom.

(D) Optimizing With Three Dimensional Geometric Shapes

- ex 7. A cylinder is inscribed in a right circular cone of altitude 12 cm and radius of base is 4 cm. Find the dimensions of the circular cylinder that will make the total surface area a maximum.

- ex 8. Find the dimensions of the circular cylinder of greatest possible volume that could be inscribed in a sphere of radius 4 cm.

(D) Optimizing With Three Dimensional Geometric Shapes

- ex 10. Naomi is designing a cylindrical can which will hold 280 mL of juice. The metal for the sides costs $0.75/m^2 while the metal for the top and bottom costs $1.40/m^2. Find the dimensions that will minimize the cost of the materials.

(E) Optimizing Revenue

- ex 9. The local orchestra wants to raise the price for season tickets for next year. Currently, there are 760 season ticket holders who paid $200 each. It is estimated that for every $10 increase in price, 20 subscribers will not buy a season ticket. What season ticket price will maximize revenue?

- ex 11. Steve is driving a truck on the highway which burns fuel at the rate of ($0.003x + 3/x$) litres per kilometer when its speed is x km/h. Fuel costs $0.65/L and Steve is paid $16/hr. What steady speed will minimize the operating costs for a 400 km trip on a highway?

(F) Further Examples

- ex 12. A north-south highway intersects an east-west highway at point P. A vehicle passes point P at 1:00 pm traveling east at a constant speed of 60 km/h. At the same instant, another vehicle is 5 km north of P, traveling south at 80 km/h. Find the time when the two vehicles are closest to each other and the distance between them at that time.

- Show how to come up with a distance equation of $d^2 = (5 - 80t)^2 + (60t)^2$

(F) Further Examples

- ex 13. I want to run a power line to a cottage being built on an island that is 400 m from the shore of a lake. The main power line ends 3 km away from the point on the shore that is closest to the island. The cost of laying the power line underwater is twice the cost of laying the power line on the land. How should I place the power line so that I minimize the overall cost?

- Show how to develop the equation $C = 2L_1 + L_2 = 2(\text{sqrt}(x^2 + 160{,}000)) + (3000 - x)$

(F) Further Examples

- ex 14. Which points on the graph of $y = 9 - x^2$ are closest to the point (0,6)?

- Show how to develop the equation $d^2 = (x - 0)^2 + (9 - x^2 - 6)^2$