SL2 - Lesson 15: Differentiation Techniques – Chain Rule | Topic 6 – Calculus

A. Lesson Context

BIG PICTURE of this UNIT:	• How do we measure "change" in a function or function model? • Why do we measure "change" in a function? • How do we analytically analyze a function or function model – beyond a simple preCalculus & visual/graphic level?		
CONTEXT of this LESSON:	Where we've been We understand how to differentiate and work with polynomial , sinusoidal, exponential & log functions	Where we are How do we differentiate and work with functions that arise from a composition of two other functions?	Where we are heading Working with more complicated functions that are variations of polynomials, sinusoidal and exp/log

B. Lesson Objectives

1. Find out how to take the derivative of a composed function.
2. Use this differentiation methods to apply calculus skills (tangents/normals & curve sketching) to simple problems in curve sketching

C. Concept Review – Composition of Functions

1. Here is a list of basic, simple functions

Recall that $(F \circ G)(x) = F(G(x))$.

Let

$$f(x) = x^2 + x + 1 \qquad g(x) = 2x^2 + 1 \qquad h(x) = \sqrt{x}$$

$$j(x) = x^{-\frac{1}{3}} \qquad k(x) = \frac{1}{x+1} \qquad m(x) = \sin(x)$$

$$p(x) = \tan(x) \qquad r(x) = \cos(x) \qquad q(x) = \sec(x)$$

2. Determine the equation of the following composite functions:

$(a)\ f \circ g(x)$ $\quad (b)\ g \circ f(x)$ $\quad (c)\ f \circ h(x)$

$(d)\ j \circ g(x)$ $\quad (e)\ h \circ k(x)$ $\quad (f)\ f \circ m(x)$

$(g)\ k \circ p(x)$ $\quad (h)\ m \circ r(x)$ $\quad (i)\ p \circ h(x)$

$(j)\ h \circ h(x)$ $\quad (k)\ j \circ p(x)$ $\quad (l)\ g \circ r(x)$

D. Skill Development – Derivatives of a Composed Functions

Now, let's use desmos.com & wolframalpha.com to develop an understanding of the derivative of a composed function.

Examples to use ➔

1. Use $y = \sin(x^3)$ which is a "new" function, made by composing $f(x) = \sin(x)$ with $g(x) = x^3$ (such that $y = f \circ g(x) = f\left(g(x)\right)$

   (a) So let's start with a prediction ➔ what do we predict the derivative of $y = \sin(x^3)$ to be? Use desmos to graph $y = \sin(x^3)$ and its derivative as well as your proposed derivative. Make observation.

   (b) Now use wolframalpha and ask wolframalpha to give us the derivative of $y = \sin(x^3)$ ➔ now how do we understand HOW that derivative came about?

2. So, now make a prediction for the derivative of $y = e^{\sin(x)}$. Use wolframalpha to confirm your prediction.

E. Skill Development – Differentiation Techniques: Summary

Rule	"formula"
Chain Rule	
Product Rule	
Quotient Rule	

F. Skill Development – Simple Practice

Let's go back to our opening exercise on composition and now take derivatives:

Recall that $(F \circ G)(x) = F(G(x))$.

Let

$$f(x) = x^2 + x + 1 \qquad g(x) = 2x^2 + 1 \qquad h(x) = \sqrt{x}$$

$$j(x) = x^{-\frac{1}{3}} \qquad k(x) = \frac{1}{x+1} \qquad m(x) = \sin(x)$$

$$p(x) = \tan(x) \qquad r(x) = \cos(x) \qquad q(x) = \sec(x)$$

1. Determine the derivative of the following composite functions:

$(a)\ f \circ g(x)$	$(b)\ g \circ f(x)$	$(c)\ f \circ h(x)$
$(d)\ j \circ g(x)$	$(e)\ h \circ k(x)$	$(f)\ f \circ m(x)$
$(g)\ k \circ p(x)$	$(h)\ m \circ r(x)$	$(i)\ p \circ h(x)$
$(j)\ h \circ h(x)$	$(k)\ j \circ p(x)$	$(l)\ g \circ r(x)$

G. Problem 2 – Applying Calculus: Working with Rates of Change & Tangents & Normals

1. (**CI**) Given the function $f(x) = \ln(2x-1)$, determine:
   i. The equation of the derivative.
   ii. The exact value of the instantaneous rate of change at $x = 2$.
   iii. At what point(s) does the function have an instantaneous rate of change of $\frac{1}{3}$?

2. (**CI**) Determine the equation of the line tangent to $y = (6x+3)^{\frac{5}{3}}$ at $x = 4$.

3. Find the equation of the line normal to the curve of $y = 4xe^{x^2-4}$ at the point $(1,4)$

4. (**CI**) Given the function $y = \sqrt{6x-5}$.
   i. State the domain for the function $y = \sqrt{6x-5}$.
   ii. Find $\frac{dy}{dx}$ for $y = \sqrt{6x-5}$.
   iii. HENCE, explain why $\frac{dy}{dx} > 0$ for all values of *x*.
   iv. What does this mean about the function?

5. (CI) Determine the equation of the tangent to $y = \cos(\pi - 2x)$ at the point where $x = \frac{\pi}{4}$.

6. (CI) Find the point(s) where the tangent line to the curve of $f(x) = e^{2x-3x^2}$ is horizontal.

7. (CI) For the curve defined by $g(x) = (\sin(x) + \cos(x))^2$ on the domain of $[0, 2\pi]$, determine:
   i. the x- and y-intercept(s);
   ii. Show that $g'(x) = 2\cos(2x)$
   iii. the first three stationary points;
   iv. the “nature” of these stationary points (max/min/neither);
   v. hence, sketch the function $g(x) = (\sin(x) + \cos(x))^2$.

8. (**CI**) Determine the interval(s) in which the curve of $g(x) = \sin^2(x)$ is concave down.

9. (**CI**) Given the function $g(x) = \cos(x) + \frac{1}{2}\cos(2x)$, $x \in [0, 2\pi]$, sketch the graph, identifying all important features including maximum(s), minimum(s) and inflection points.