Lesson 5 - IRoC, Derivatives and First Principles

SL2 – Topic 6 Calculus - Santowski

Lesson Objectives

1. Calculate an instantaneous rate of change using first principles ➔ difference quotients and limits
2. Introduce calculus related terminology
3. Work with tangent slopes & make the connection to instantaneous rates of change, tangents and derivatives at a point
4. Calculate instantaneous rates of change in real world problems

Fast Five

- 1. Find an expression for the slope of a line segment passing through the points (x, f(x)) and (x+h, f(x+h)) if $f(x) = x^2 - 2x - 5$
- 2. Find the slope of the line passing through the point (2,-5), if this line is to be tangent to the function $f(x) = x^2 - 2x - 5$

(A) Slopes of Tangents ➔ Meaning

- Recall what slopes mean ➔ slope means a rate of change (the change in "rise" given the change in "run")
- Recall the slope formulas:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

$$m = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

(A) Slopes of Tangents ➔ Meaning

- So finding the slope of a tangent line to a curve at a point is the same as finding the rate of change of the curve at that point
- We call this the INSTANTANEOUS rate of change of the function at the given point

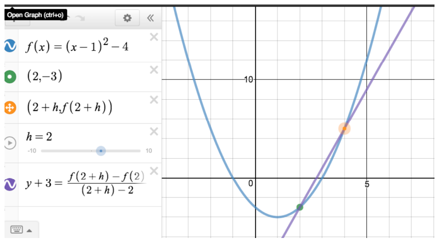
(B) Instantaneous Rates of Change - Algebraic Calculation

- We have calculated the tangent slope at the point where $x = a$ using formula as follows (to create a more "algebra friendly" formula)

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- See the diagram on the next page for an explanation of the formula

(C) Instantaneous Rates of Change – Graphic Representation

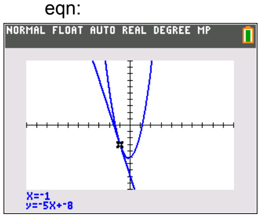
- 

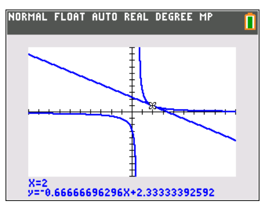
(D) Example - Determine the Slope of a Tangent Line & IRoC

- Determine the slope of the tangent line to the curve $f(x) = 3x^2 + x - 5$ at the point (-1,-3)
- Alternate way to ask a similar question:
- Determine the instantaneous rate of change of the function $f(x) = \frac{3}{2x-1}$ at the point (2 ,1)

(D) Example - Determine the Slope of a Tangent Line & IRoC

- Now let's confirm this using the TI-84
- So ask the calculator to graph and determine the tangent eqn:





(E) New Terminology

- The phrase "***derivative of a function at a point***": means the SAME as:
- (1) instantaneous rate of change of the function at a point;
- (2) the slope of the curve at that point
- (3) the slope of the line that is tangent to the curve at that point
- The **process** of finding a derivative of a function is called **differentiation**.

(E) New Terminology

- The **process** of finding a derivative of a function is called **differentiation**.
- The process of differentiation ***using our limiting process*** (and using the limit formula) is referred to as "differentiation using the ***method of first principles***"

(F) New Notations

- The "***derivative of a function at a point***" is denoted by:

(i) $f'(x)$ or $f'(a)$

(ii) y' or $y'(a)$

(iii) $\frac{dy}{dx}$

(iv) $\frac{d}{dx}f(x)$ or $\frac{d}{dx}f(a)$

(v) $\left.\frac{dy}{dx}\right|_{x=a}$

(G) Further Examples - Determine the Slope of a Tangent Line

- Make use of the limit definition of a derivative (first principles) to determine the instantaneous rate of change of the following functions at the given *x* values:

$$f(x) = 2x^2 - 5x + 40 \quad at \quad x = 4$$
$$g(x) = 1 - 3x^2 \quad at \quad x = -1$$
$$h(x) = x^3 - x \quad at \quad x = 1$$

(H) Applications - Determine the Slope of a Tangent Line

- A football is kicked and its height is modeled by the equation h(t) = -4.9t² + 16t + 1, where h is height measured in meters and t is time in seconds. Determine the instantaneous rate of change of height at 1s, 2s, 3s using first principles.
- Confirm with technology.

(H) Applications - Determine the Slope of a Tangent Line

- The solution is:

$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t \to 0} \frac{h(1+\Delta t) - h(1)}{\Delta t}$$
$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t \to 0} \frac{\left(-4.9(1+\Delta t)^2 + 16(1+\Delta t) + 1\right) - \left(-4.9(1)^2 + 16(1) + 1\right)}{\Delta t}$$
$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t \to 0} \frac{\left(-4.9\Delta t^2 + 6.2\Delta t + 12.1\right) - (12.1)}{\Delta t}$$
$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t \to 0} (-4.9\Delta t + 6.2)$$
$$m = \left.\frac{dh}{dt}\right|_{t=1} = 6.2$$

(H) Applications - Determine the Slope of a Tangent Line

- So the slope of the tangent line (or the instantaneous rate of change of height) is 6.2 ➔ so, in context, the rate of change of a distance is called a speed (or velocity), which in this case would be 6.2 m/s at t = 1 sec. ➔ now simply repeat, but use t = 2,3 rather than 1

(H) Applications - Determine the Slope of a Tangent Line

- A business estimates its profit function by the formula P(x) = x³ - 2x + 2 where x is millions of units produced and P(x) is in billions of dollars. Determine the value of the tangent slope at x = ½ and at x = 1½ using the method of first principles. How would you interpret these values (that correspond to tangent slopes)?

(I) Examples

- Find the value of derivative of y = -2(x+1)(x-3) at the point P(2,6).
- Find the equation of the line tangent to $y = 2\sqrt{x+3}$ at the point P(6,6).
- Find the equation of the line orthogonal to $y = \frac{2}{x-4}$ at the point P(2,-1).