Lesson 4 - Limits & Instantaneous Rates of Change

SL2 – Topic 6 Calculus - Santowski

Lesson Objectives

1. Calculate an instantaneous rate of change using difference quotients and limits
2. Calculate instantaneous rates of change numerically, graphically, and algebraically
3. Calculate instantaneous rates of change in real world problems
4. Work with tangent slopes & make the connection to instantaneous rates of change

Fast Five

- 1. State the slope of the line perpendicular to the line 2x – 3y = 6
- 2. Find the equation of a line passing through (-3,5) and (4,-5)
- 3. A linear function is given by $y = \frac{1-x}{2}$. If x increases by 4, by how much does y change?
- 4. Find an expression for the slope of a line passing through (x, f(x)) and (x+h, f(x+h))

(A) Opening Exercise: Average Rates of Change: Motion

- If a ball is thrown into the air with a velocity of 30 m/s, its height, *h*, in meters after *t* seconds is given by $h(t)=30t-5t^2$.
- Determine the distance traveled in the 3rd second.
- Find the average velocity for the time period beginning when $t = 2$ sec and lasting:
  - (i) 1 s
  - (ii) 0.5 s
  - (iii) 0.1 s
- So, now estimate the instantaneous velocity at $t = 2$

(B) Slopes of Tangents

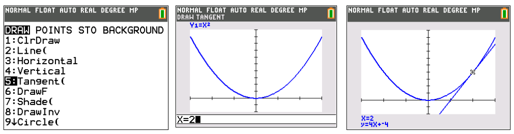
- Now we want to take this concept of rates of change one step further and develop a process whereby we can "estimate" instantaneous rates of change and tangent line slopes

(B) Slopes of Tangents

- Given the function $f(x) = x^2$, HOW WOULD YOU GRAPHICALLY (using TI-84) determine the slope of the tangent line to the curve at the point (1,1).
- What would your answer mean?

(B) Slopes of Tangents – Graphically on TI-84

- Here is how we can visualize (and solve) the problem using the GDC ➔ We simply ask the calculator to graph the tangent line!!



(B) Slopes of Tangents ➔ Meaning

- So finding the SLOPE OF A TANGENT line to a curve at a point is the same as finding the rate of change of the curve at that point
- We call this the INSTANTANEOUS rate of change of the function at the given point

(B) Slopes of Tangents

- Given the function $f(x) = x^2$, HOW WOULD YOU ALGEBRAICALLY determine the EXACT slope of the tangent line to the curve at the point (1,1).
- What would your answer mean?

(B) Slopes of Tangents ➔ Meaning

- Recall what slopes mean ➔ slope means a rate of change (the change in "rise" given the change in "run")
- Recall the slope formulas:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

$$m = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

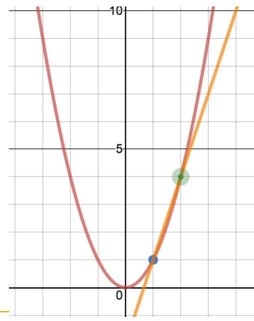
(B) Estimating Tangent Slopes – Using Secant Slopes

- For the time being we cannot directly use our algebra skills to find the slope of a line if we only know ONE point on the line
- STRATEGY: Let's use secant slopes and make estimates using a secant slope!

(C) Estimating Tangent Slopes: Using Secant Slopes

- Given $f(x) = x^2$, use the TI-84 to estimate the slope of the tangent line to the curve at the point (1,1)
- If Q is a second point on the curve, find the slope of the secant PQ if the x-coordinate of Q is:
  - (i) x = 2 (ii) x = 1.5
  - (iii) x = 1.1 (iv) x = 1.01
  - (v) x = 1.001 (vi) x = 1 + h

https://www.desmos.com/calculator/bphiooejd8



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(C) Estimating Tangent Slopes: Using Secant Slopes

- If Q is a second point on the curve, find the slope of the secant PQ if the x-coordinate of Q is

$$m = \frac{f(1+h) - f(1)}{(1+h) - (1)}$$
$$m = \frac{(1+h)^2 - (1)^2}{h}$$
$$m = \frac{\left(1 + 2h + h^2\right) - (1)}{h}$$
$$m = \frac{h^2 + 2h}{h}$$
$$m = \frac{\cancel{h}(h+2)}{\cancel{h}}$$
$$m = h + 2$$

(C) Estimating Tangent Slopes: Using Secant Slopes

- Given f(x) = x^2, use the TI-84 to "estimate" the slope of the tangent line to the curve at the point (1,1)
- If the second point on the curve is at x = 1 + h, the slope of the secant will be given m_{sec} = 2 + h
- Recall the "significance" of h
- So how would we now "estimate" the slope of the tangent at x = 2?

(C) Estimating Tangent Slopes: Using Secant Slopes

- Given f(x) = 4x – x^2, we will "estimate" the slope of the tangent line to the curve at the point (1,3)
- If the second point on the curve is now 1 + h, find the slope of the secant PQ
- Now predict the slope of the tangent line.

(C) Estimating Tangent Slopes: Using Secant Slopes

- $P\left(1, \frac{1}{2}\right)$ lies on the curve $y = \frac{1}{x+1}$
- If the second point on the curve is 1 + h, find the slope of the secant PQ
- Now predict the slope of the line tangent to $y = \frac{1}{x+1}$ at x = 1

(C) Estimating Tangent Slopes: Using Secant Slopes

- Given the function f(x) = x^3 + x – 2, "estimate" the slope of the tangent line to the curve at the point (2,8)
- If the second point on the curve is 2 + h, find the slope of the secant line and thus "estimate" the slope of the tangent line at x = 2

(E) Instantaneous Rates of Change - Algebraic Calculation

- We will adjust the tangent slope formula as follows (to create a more "algebra friendly" formula)

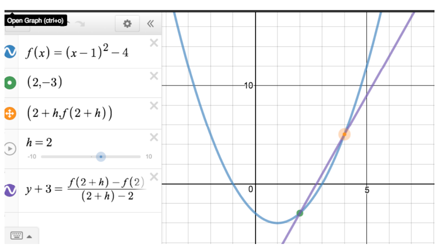
$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- See the diagram on the next page for an explanation of the formula

(E) Instantaneous Rates of Change - Algebraic Calculation

- Given the function f(x) = (x – 1)^2 – 4, determine the slope of the curve at the point (2,-3).
- Let Point Q be (2 + h, f(2 + h))
- Visualize this process on DESMOS

(E) Instantaneous Rates of Change - Algebraic Calculation

- 

(E) Example - Determine the Slope of a Curve

- $$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{(2+h) - (2)}$$
$$m = \lim_{h \to 0} \frac{\left(\left((2+h)-1\right)^2 - 4\right) - \left((2-1)^2 - 4\right)}{h}$$
$$m = \lim_{h \to 0} \frac{\left((1+h)^2 - 4\right) - (-3)}{h}$$
$$m = \lim_{h \to 0} \frac{\left(h^2 + 2h + 1 - 4\right) - (-3)}{h}$$
$$m = \lim_{h \to 0} \frac{\cancel{h}(h+2)}{\cancel{h}}$$
$$m = \lim_{h \to 0} \left(-\cancel{h} + 2\right)$$
$$m = 2$$

(E) Example - Determine the Slope of a Tangent Line

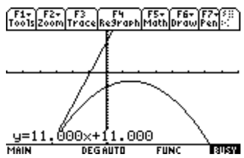
- Determine the slope of the tangent line to the curve f(x) = -x^2 + 3x - 5 at the point (-4,-33)
- Alternate way to ask the same question:
- Determine the instantaneous rate of change of f(x) = -x^2 + 3x - 5 at the point (-4,-33)

(E) Example - Determine the Slope of a Tangent Line

- $$m = \lim_{h \to 0} \frac{f(-4+h) - f(-4)}{h}$$
$$m = \lim_{h \to 0} \frac{\left(-(-4+h)^2 + 3(-4+h) - 5\right) - (-33)}{h}$$
$$m = \lim_{h \to 0} \frac{\left(-\left(h^2 - 8h + 16\right) + (-12 + 3h) - 5\right) - (-33)}{h}$$
$$m = \lim_{h \to 0} \frac{\left(-h^2 + 11h - 33\right) - (-33)}{h}$$
$$m = \lim_{h \to 0} \frac{\cancel{h}(-h+11)}{\cancel{h}}$$
$$m = \lim_{h \to 0} \left(-\cancel{h} + 11\right)$$
$$m = 11$$

(E) Example - Determine the Slope of a Tangent Line

- Now let's confirm this using the graphing features and the calculus features of the TI-84
- So ask the calculator to graph and determine the tangent eqn:



(F) Further Examples - Determine the Slope of a Tangent Line

- Make use of the limit definition of an instantaneous rate of change to determine the instantaneous rate of change of the following functions at the given *x* values:

$$f(x) = 2x^2 - 5x + 40 \quad at \quad x = 4$$
$$g(x) = 1 - 3x^2 \quad at \quad x = -1$$
$$h(x) = x^3 - x \quad at \quad x = 1$$

(G) Applications - Determine the Slope of a Tangent Line

- A football is kicked and its height is modeled by the equation h(t) = -4.9t² + 16t + 1, where h is height measured in meters and t is time in seconds. Determine the instantaneous rate of change of height at 1 s, 2 s, 3 s.

(G) Applications - Determine the Slope of a Tangent Line

- The solution is:

$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t \to 0} \frac{h(1+\Delta t) - h(1)}{\Delta t}$$
$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t \to 0} \frac{\left(-4.9(1+\Delta t)^2 + 16(1+\Delta t) + 1\right) - \left(-4.9(1)^2 + 16(1) + 1\right)}{\Delta t}$$
$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t \to 0} \frac{\left(-4.9\Delta t^2 + 6.2\Delta t + 12.1\right) - (12.1)}{\Delta t}$$
$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t \to 0}\left(-4.9\Delta t + 6.2\right)$$
$$m = \left.\frac{dh}{dt}\right|_{t=1} = 6.2$$

(G) Applications - Determine the Slope of a Tangent Line

- So the slope of the tangent line (or the instantaneous rate of change of height) is 6.2 ➔ so, in context, the rate of change of a distance is called a speed (or velocity), which in this case would be 6.2 m/s at t = 1 sec. ➔ now simply repeat, but use t = 2,3 rather than 1

(G) Applications - Determine the Slope of a Tangent Line

- An object moves along a straight line. Its position, *x* meters (from a fixed point O), at time, *t* seconds is given by $x(t) = t - \frac{1}{4}t^2,\ t \geq 0$
- Determine the instantaneous velocity at *t* = 1

(G) Applications - Determine the Slope of a Tangent Line

- A business estimates its profit function by the formula $P(x) = x^3 - 2x + 2$ where x is millions of units produced and P(x) is in billions of dollars. Determine the value of the tangent slope at x = ½ and at x = 1½. How would you interpret these values (that correspond to tangent slopes)?

(G) Applications - Determine the Slope of a Tangent Line

- The population of a city at the start of 2000 was 2.3 million and its projected population, *N* million, is modeled by the equation given below, where $t \geq 0$ and is measured in years since the beginning of 2000.

$$N(t) = 2.3e^{0.0142t},\ t \geq 0$$

  - (a) Determine the average annual change of the population.
  - (b) Use the TI-84 & idea of "local linearity" to estimate the growth of the city at the beginning of 2005 (instantaneous RoC at $t = 5$)
  - (c) Use the TI-84 & draw the tangent line at t = 5 to determine the the growth of the city at the beginning of 2005
  - (d) Now try it algebraically …….. ? ? ? ?