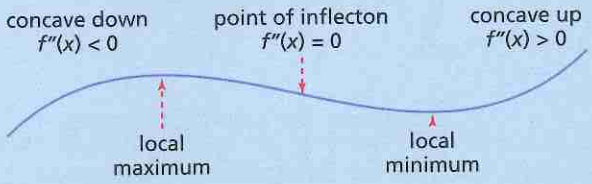
KEY IDEAS

- The graph of a function $f(x)$ is **concave up** on an interval if f' is increasing on the interval. The graph of a function $f(x)$ is **concave down** on an interval if $f'(x)$ is decreasing.
- **Test for Concavity:** Let f be a differentiable function whose second derivative exists on an open interval I.
  - The graph of $f(x)$ is concave up if $f''(x) > 0$ for all x in I.
  - The graph of $f(x)$ is concave down if $f''(x) < 0$ for all x in I.

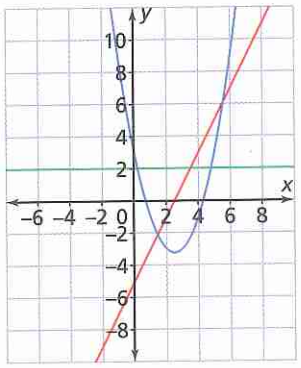


- **Point of Inflection:** A point of inflection is a point on the graph of f where the function changes from concave up to concave down or vice versa. $f''(c) = 0$ if $(c, f(c))$ is a point of inflection on the graph of a polynomial function $f(x)$ and $f''(c)$ exists.
- **The Second Derivative Test:** Suppose $f(x)$ is a function where $f'(c) = 0$ and the second derivative of $f(x)$ exists on an interval containing c.
  - If $f''(c) > 0$, then $f(c)$ is a local minimum value.
  - If $f''(c) < 0$, then $f(c)$ is a local maximum value.
  - If $f''(c) = 0$, then the test fails. Use the first derivative test in this case.

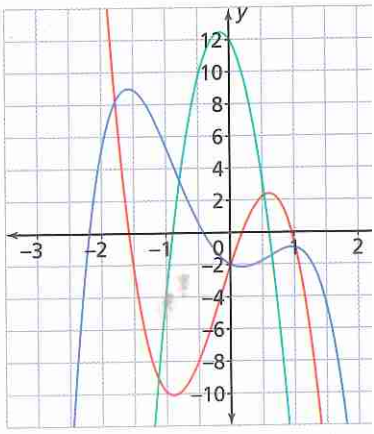
4.7 Exercises

A 1. Each figure includes the graphs of $f(x)$, $f'(x)$, and $f''(x)$. Which curve is which?

(a)

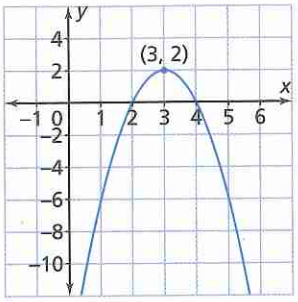


(b)

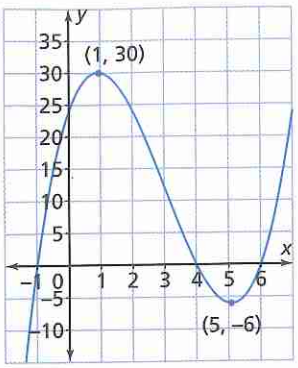


2. Graph $f'(x)$ and $f''(x)$ given the graph of $f(x)$.

(a)



(b)



3. Determine the second derivative.

(a) $f(x) = 6x^2 - 8x + 5$

(b) $f(x) = -4x^3 - 9x^2 + 12x - 3$

(c) $y = 5x^4 - 7x^2 + 23x$

(d) $g(x) = 8x^6 + 4x^4 - 3x + 4x - 11$

(e) $y = (2x + 5)(3x - 9)$

(f) $h(x) = (2x - 5)(4x^2 + 10x + 25)$

4. The second derivative of $f(x)$ is given. Determine the intervals of concavity and the points of inflection. Copy and complete the table.

(a) $f''(x) = 3(x - 4)(x + 2)$

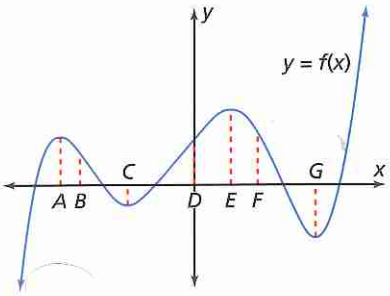
	Intervals		
	$x < -2$	$-2 < x < 4$	$x > 4$
$3(x - 4)$			
$(x + 2)$			
$f''(x)$			
Concavity in the graph of $f(x)$			

(b) $f''(x) = -(2x + 1)(x - 3)(x - 5)$

	Intervals			
	$x < -0.5$	$-0.5 < x < 3$	$3 < x < 5$	$x > 5$
$-(2x + 1)$				
$(x - 3)$				
$(x - 5)$				
$f''(x)$				
Concavity in the graph of $f(x)$				

5. For the graph shown, identify the points or intervals where each derivative is positive, negative, and 0.

(a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$



B

6. Find the point of inflection on the curve defined by $y = x^3 + 5$. Show that the tangent line at this point crosses the curve.

7. Determine the intervals of concavity and the points of inflection.

(a) $y = 6x - 2x^2$

(b) $y = 3x^2 + 5x - 1$

(c) $y = 3x^2 - 8x$

(d) $y = x^4 - 6x^2$

(e) $y = x^4 + 4x^3 - 18x^2$

(f) $y = x^4 - 6x^3 + 10$

8. Determine the intervals where the graph of the function is rising or falling, the intervals where the graph is concave up or concave down, the maximum and minimum points, and the points of inflection. Use the information to graph the function.

(a) $y = x^2 + 6x - 10$

(b) $y = 3x^2 - x^3$

(c) $y = x^3 - 3x - 4$

(d) $y = x^3 - 6x^2 + 12x - 8$

(e) $y = 2x^4 - 8x + 2$

(f) $y = \frac{1}{3}x^3 - 9x + 3$

(g) $y = \frac{1}{4}x^2 - 2x^2$

(h) $y = x^4 + 6x^3 - 24x^2 + 26$

(i) $y = x(x - 4)^3$

9. Verify your results for question 8 using graphing technology.

10. **Knowledge and Understanding:** Determine the intervals of concavity and the points of inflection for the graph of $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$.

11. Graph part of a polynomial function that is

    (a) concave down and falling
    (b) concave up and rising
    (c) concave down and rising
    (d) concave up and falling

12. Graph the function $f(x)$ for which

    (a) $f(3) = f(5) = 0$
    $f'(x) < 0$ if $x < 4$
    $f'(4) = 0$
    $f'(x) > 0$ if $x > 4$
    $f''(x) < 0$ for all x

    (b) $f'(x) > 0$ if $x < -1$
    $f'(x) > 0$ if $x > 3$
    $f'(x) < 0$ if $-1 < x < 3$
    $f''(x) < 0$ if $x < 2$
    $f''(x) > 0$ if $x > 2$

13. **Communication:** Graph the continuous function $f(x)$ for which

    (a) $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$
    (b) $f'(x) > 0$ for $x < -1$ and $x > 1$; $f'(x) < 0$ for $-1 < x < 1$

14. **Application:** Show that the point of inflection in the graph of $g(x) = x(x - 6)^2$ lies halfway between the local maximum and minimum points. Is the point of inflection always at the midpoint between the local maximum and minimum points?

15. Determine all extrema of the function. Use the second derivative test to verify, where possible.

    (a) $y = -2x^2 + 12x - 2$
    (b) $y = x^2 + 5x - 10$
    (c) $y = x^5 - 5x^3$
    (d) $y = 3x^2 - x^3$
    (e) $y = x^4 + 8x^3 + 18x^2$
    (f) $y = x^2(2 - x)^2$

16. A manufacturer estimates that it costs $x^3 - 12x^2 + 400x + 600$ dollars to produce x thousand units. What production level will minimize marginal cost?

17. **Thinking, Inquiry, Problem Solving:** Find the inflection points, if any exist, for the graph of $f(x) = (x - c)^n$, for $n = 1, 2, 3$, and 4. What conclusion can you draw about the value of n and the existence of inflection points on the graph of f?

18. **Check Your Understanding:** Can a point of inflection also be a local maximum point or a local minimum point? Explain.

C 19. For $y = x^4 - kx^3 + 10$, find the value of k such that the difference between the x-coordinates of the two inflection points is 2.

20. Prove that the graph of a cubic polynomial function has exactly one inflection point.

21. Prove that the graph of a cubic function with three distinct, real zeros has a point of inflection whose x-coordinate is the average of the three zeros.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: For $f(x) = 2x^3 - 9x^2 + 12x - 2$,

(a) determine the intervals of increase and decrease

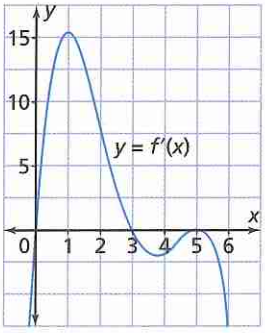
(b) determine the local maximum and minimum values

(c) determine the intervals of concavity

(d) determine any inflection points

(e) graph the function

Application: The position, s, of an object moving in a straight line from a fixe point at t seconds is given by $s(t) = t^3 - 9t^2 - 21t - 11$, where $t > 0$ and $s($ is measured in metres.



(a) Determine the times when the object is stopped and when it is moving forward. Is the acceleration negative at any time?

(b) At 2 s, is the object speeding up or slowing down? Explain why.

Thinking, Inquiry, Problem Solving: For what values of the constants c and d $(4, -7)$ a point of inflection on the cubic curve $y = x^3 + cx^2 + x + d$?

Communication: Given the graph of $y = f'(x)$, explain where the graph of $y = f(x)$ has a maximum point, a minimum point, or a point of inflection.

The Chapter Problem

Trends in Post-Secondary Education

Apply what you learned in this section to answer these questions abou The Chapter Problem on page 264.

CP8. Determine the second derivative of the mathematical model. U the second derivative to locate where the graph of the functior concave up and concave down between 1980 and 1998.

CP9. Determine all points of inflection for the graph of the mathematical model.

CP10. Use the second derivative test to verify the existence of the extrema.

CP11. Use all the information you have obtained about the mathematical model to graph it.