Lesson 92 – Poisson Distribution

HL MATH - SANTOWSKI

The Poisson Experiment

Used to find the probability of a rare event

- Randomly occurring in a specified interval
  - – Time
  - – Distance
  - – Volume
- Measure number of rare events occurred in the specified interval

The Poisson Experiment

Counting the number of times a success occur in an interval

- Probability of success the same for all intervals of equal size
- Number of successes in interval independent of number of successes in other intervals
- Probability of success is proportional to the size of the interval
- Intervals do not overlap

The Poisson Experiment

- Some of the properties for a Poisson experiment can be hard to identify.
- Generally only have to consider whether:-

– The experiment is concerned with counting number of RARE events occur in a specified interval

– These RARE events occur randomly

– Intervals within which RARE event occurs are independent and do not overlap

The Poisson Experiment

Typical cases where the Poisson experiment applies:

– Accidents in a day in Cairo

– Bacteria in a liter of water

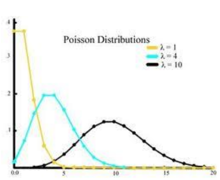
– Dents per square meter on the body of a car

– Viruses on a computer per week

– Complaints of mishandled baggage per 1000 passengers

Application:

The Poisson distribution arises in two ways:



1. As an approximation to the binomial when p is small and n is large:

Example: In auditing when examining accounts for errors; n, the sample size, is usually large. p, the error rate, is usually small.

2. Events distributed independently of one another in time:

X = the number of events occurring in a fixed time interval has a Poisson distribution.

Example: X = the number of telephone calls in an hour.

$$PDF: p(x) = e^{-\lambda}\frac{\lambda^x}{x!},\ x = 0, 1, 2, \cdots; \lambda > 0$$

What is Poisson distribution?

The Poisson distribution is a discrete probability distribution for the counts of events that occur randomly in a given interval of time (or space).

Many experimental situations occur in which we observe the counts of events within a set unit of time, area, volume, length etc.

Examples:

The number of cases of a disease in different towns

The number of mutations in set sized regions of a chromosome

Applications

the number of deaths by horse kicking in the Prussian army (first application)

birth defects and genetic mutations

rare diseases (like Leukaemia, but not AIDS because it is infectious and so not independent) - especially in legal cases

car accidents

traffic flow and ideal gap distance

number of typing errors on a page

hairs found in McDonald's hamburgers

spread of an endangered animal in Africa

failure of a machine in one month

Poisson or not?

Which of the following are likely to be well modelled by a Poisson distribution?

(can select more than one)

1. Number of duds found when I test four components
2. The number of heart attacks in Brighton each year
3. The number of planes landing at Heathrow between 8 and 9am
4. The number of cars getting punctures on the M1 each year
5. Number of people in the UK flooded out of their home in July

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Are they Poisson? *Answers*:

Number of duds found when I test four components

- NO: this is Binomial
(it is not the number of independent random events in a continuous interval)

The number of heart attacks in Brighton each year

- YES: large population, no obvious correlations between heart attacks in different people

The number of planes landing at Heathrow between 8 and 9am

- NO: 8-9am is rush hour, planes land **regularly** to land as many as possible (1-2 a minute) – they do not land at random times or they would hit each other!

The number of cars getting punctures on the M1 each year

- YES (roughly): If punctures are due to tires randomly wearing thin, then expect punctures to happen independently at random
But: may not all be independent, e.g. if there is broken glass in one lane

Number of people in the UK flooded out of their home in July

- NO: floodings of different homes not at all independent; usually a small number of floods each flood many homes at once, $P(\text{flooded}|\text{next door flooded}) \gg P(\text{flooded})$

Equation

If X = The number of events in a given interval,

Then, if the mean number of events per interval is λ.

The probability of observing x events in a given interval is given by,

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!} \qquad x = 0, 1, 2, 3, 4, \ldots$$

e is a mathematical constant. e≈2.718282.

Problem number 1:

Consider, in an office 2 customers arrived today. Calculate the possibilities for exactly 3 customers to be arrived on tomorrow.

Solution

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Solution

Step1: Find $e^{-\lambda}$.
Where , $\lambda=2$ and $e=2.718$
$e^{-\lambda} = (2.718)^{-2} = 0.135$.

Step2: Find λ^{x}.
where, $\lambda=2$ and $x=3$.
$\lambda^{x} = 2^3 = 8$.

Step3: Find f(x).
$P(X=x) = e^{-\lambda}\lambda^{x} / x!$
$P(X=3) = (0.135)(8) / 3! = 0.18$.

Hence there are 18% possibilities for 3 customers to be arrived on tomorrow.

Problem number 2:

Births in a hospital occur randomly at an average rate of 1.8 births per hour.

i) What is the probability of observing 4 births in a given hour at the hospital?

ii) What about the probability of observing more than or equal to 2 births in a given hour at the hospital?

Solution:

Problem number 2:

Births in a hospital occur randomly at an average rate of 1.8 births per hour. i) What is the probability of observing 4 births in a given hour at the hospital? ii) What about the probability of observing more than or equal to 2 births in a given hour at the hospital?

Solution:

Let X = No. of births in a given hour

i) $P(X=4) = e^{-1.8}\frac{1.8^4}{4!} = 0.0723$

ii) More than or equal to 2 births in a given hour

$$
\begin{aligned}
P(X \geq 2) &= P(X=2) + P(X=3) + \ldots \\
&= 1 - P(X<2) \\
&= 1 - (P(X=0) + P(X=1)) \\
&= 1 - (e^{-1.8}\frac{1.8^0}{0!} + e^{-1.8}\frac{1.8^1}{1!}) \\
&= 1 - (0.16529 + 0.29753) \\
&= 0.537
\end{aligned}
$$

Problem number 3.

Consider a computer system with Poisson job-arrival stream at an average of 2 per minute. Determine the probability that in any one-minute interval there will be

(i) 0 jobs;
(ii) exactly 2 jobs;
(iii) at most 3 arrivals.

Solution

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Solution

(i) No job arrivals: $P(X = 0) = e^{-2} = .135$

(ii) Exactly 3 job arrivals: $P(X = 3) = e^{-2}\frac{2^3}{3!} = .18$

(iii) At most 3 arrivals

$$\begin{aligned} P(X \leq 3) &= P(0) + P(1) + P(2) + P(3) \\ &= e^{-2} + e^{-2}\frac{2}{1} + e^{-2}\frac{2^2}{2!} + e^{-2}\frac{2^3}{3!} \\ &= 0.1353 + 0.2707 + 0.2707 + 0.1805 \\ &= 0.8571 \end{aligned}$$

Problem number:4

Given that 5% of a population are left-handed, use the Poisson distribution to estimate the probability that a random sample of 100 people contains 2 or more left-handed people.

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X = No. of left handed people in a sample of 100

$X \sim \text{Bin}(100, 0.05)$

Poisson approximation $\Rightarrow X \sim \text{Po}(\lambda)$ with $\lambda = 100 \times 0.05 = 5$

We want $P(X \geq 2)$?

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) \\ &= 1 - \Big(P(X = 0) + P(X = 1)\Big) \\ &\approx 1 - \left(\mathrm{e}^{-5}\frac{5^0}{0!} + \mathrm{e}^{-5}\frac{5^1}{1!}\right) \\ &\approx 1 - 0.040428 \\ &\approx 0.959572 \end{aligned}$$

Example: On average lightning kills three people each year in the UK, $\lambda = 3$. What is the probability that only one person is killed this year?

Answer:

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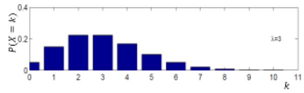
Answer:

Assuming these are independent random events, the number of people killed in a given year therefore has a Poisson distribution:



Let the random variable X be the number of people killed in a year.

Poisson distribution $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$ with $\lambda = 3$

$$\Rightarrow P(X = 1) = \frac{e^{-3}3^1}{1!} \approx 0.15$$

Poisson distribution

Suppose that trucks arrive at a receiving dock with an average arrival rate of 3 per hour. What is the probability exactly 5 trucks will arrive in a two-hour period?

Q

1. $\frac{e^{-3}3^5}{5!}$
2. $\frac{e^{-3}3^{2.5}}{2.5!}$
3. $\frac{e^{-5}5^6}{6!}$
4. $\frac{e^{-6}6^5}{5!}$

Reminder: $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$

Q

Poisson distribution

Suppose that trucks arrive at a receiving dock with an average arrival rate of 3 per hour. What is the probability exactly 5 trucks will arrive in a two-hour period?

A

In two hours mean number is $\lambda = 2 \times 3 = 6$.

$$P(X = k = 5) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-6}6^5}{5!}$$

***Example:* Telecommunications**

Messages arrive at a switching centre at random and at an average rate of 1.2 per second.

(a) Find the probability of 5 messages arriving in a 2-sec interval.

(b) For how long can the operation of the centre be interrupted, if the probability of losing one or more messages is to be no more than 0.05?

***Example:* Telecommunications**

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(a) Find the probability of 5 messages arriving in a 2-sec interval.

(b) For how long can the operation of the centre be interrupted, if the probability of losing one or more messages is to be no more than 0.05?

Answer:

Times of arrivals form a Poisson process, rate $v = 1.2/\text{sec}$.

(a) Let Y = number of messages arriving in a 2-sec interval.

Then Y ~ Poisson, mean number $\lambda = vt = 1.2 \times 2 = 2.4$

$$P(Y = k = 5) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-2.4}2.4^5}{5!} = 0.060$$

Question: (b) For how long can the operation of the centre be interrupted, if the probability of losing one or more messages is to be no more than 0.05?

Answer:

(b) Let the required time = t seconds. Average rate of arrival is 1.2/second.

Let k = number of messages in t seconds, so that

$k \sim$ Poisson, with $\lambda = 1.2 \times t = 1.2t$

Want P(At least one message) = $P(k \geq 1) = 1 - P(k = 0) \leq 0.05$

$$P(k = 0) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-1.2t}(1.2t)^0}{0!} = e^{-1.2t}$$

$\Rightarrow \mathbf{1 - e^{-1.2t} \leq 0.05}$

$\Rightarrow \quad -e^{-1.2t} \leq 0.05 - 1$

$\Rightarrow e^{-1.2t} \geq 0.95$

$\Rightarrow -1.2t \geq \ln(0.95) = -0.05129$

$\boxed{\Rightarrow t \leq 0.043 \text{ seconds}}$

Example

The probability of a certain part failing within ten years is 10^{-6}. Five million of the parts have been sold so far.

What is the probability that three or more will fail within ten years?

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Answer:

Let X = number failing in ten years, out of 5,000,000; $X \sim B(5000000, 10^{-6})$

Evaluating the Binomial probabilities is rather awkward; better to use the Poisson approximation.

X has approximately Poisson distribution with $\lambda = np = 5000000 \times 10^{-6} = 5$.

$$\text{P(Three or more fail)} = P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2)$$
$$= 1 - \frac{e^{-5}5^0}{0!} - \frac{e^{-5}5^1}{1!} - \frac{e^{-5}5^2}{2!}$$
$$= 1 - e^{-5}(1 + 5 + 12.5) = 0.875$$

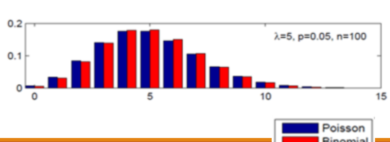
For such small p and large n the Poisson approximation is very accurate (exact result is also 0.875 to three significant figures).

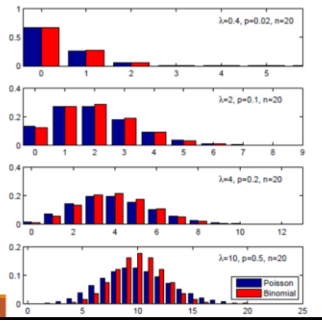
Approximation to the Binomial distribution

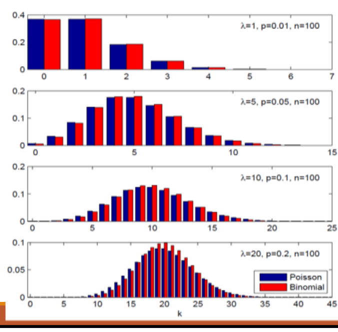
The Poisson distribution is an approximation to B(n, p), when n is large and p is small (e.g. if $np < 7$, say).

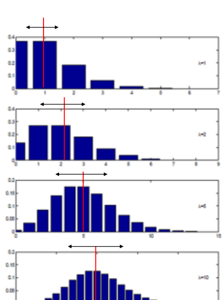
In that case, if $X \sim B(n,p)$ then $P(X=k) \approx \frac{e^{-\lambda}\lambda^k}{k!}$ Where $\lambda = np$

i.e. X is approximately Poisson, with mean $\lambda = np$.









Mean and variance

If X ~ Poisson with mean λ, then

$$\mu = E(X) = \lambda$$

$$\sigma^2 = \text{var}(X) = \lambda$$

Poisson Distribution Summary

Describes discrete random variable that is the number of *independent* and *randomly occurring* events, with mean number λ. Probability of k such events is

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Mean and variance: $\mu = \sigma^2 = \lambda$

The sum of Poisson variables $\sum X_i$ is also Poisson, with average number $\sum_i \lambda_i$

Approximation to Binomial for large n and small p:

if $X \sim B(n, p)$ then $P(X = k) \approx \frac{e^{-\lambda}\lambda^k}{k!}$ where $\lambda = np$