Lesson 88 – Probability with Combinatorics

HL2 Math - Santowski

Fundamental Counting Principle

Fundamental Counting Principle can be used determine the number of possible outcomes when there are two or more characteristics .

Fundamental Counting Principle states that if an event has m possible outcomes and another independent event has n possible outcomes, then there are $m * n$ possible outcomes for the two events together.

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Lets start with a simple example.

A student is to roll a die and flip a coin. How many possible outcomes will there be?

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1H	2H	3H	4H	5H	6H
1T	2T	3T	4T	5T	6T

12 outcomes

6*2 = 12 outcomes

Fundamental Counting Principle

For a college interview, Robert has to choose what to wear from the following: 4 slacks, 3 shirts, 2 shoes and 5 ties. How many possible outfits does he have to choose from?

Fundamental Counting Principle

For a college interview, Robert has to choose what to wear from the following: 4 slacks, 3 shirts, 2 shoes and 5 ties. How many possible outfits does he have to choose from?

4*3*2*5 = 120 outfits

Permutations

A **Permutation** is an arrangement of items in a particular order.

Notice, ORDER MATTERS!

To find the number of Permutations of n items, we can use the Fundamental Counting Principle or factorial notation.

Permutations

The number of ways to arrange the letters ABC: ___ ___ ___

Number of choices for first blank? 3 ___ ___

Number of choices for second blank? 3 2 ___

Number of choices for third blank? 3 2 1

3*2*1 = 6 3! = 3*2*1 = 6

ABC ACB BAC BCA CAB CBA

Permutations

To find the number of Permutations of n items chosen r at a time, you can use the formula

$_5P_3$

Permutations

To find the number of Permutations of n items chosen r at a time, you can use the formula

$$_{n}p_{r} = \frac{n!}{(n-r)!} \quad \text{where } 0 \le r \le n.$$

$$_{5}p_{3} = \frac{\mathbf{5!}}{\mathbf{(5-3)!}} = \frac{5!}{2!} = \mathbf{5*4*3=60}$$

Permutations

Practice:

A combination lock will open when the right choice of three numbers (from 1 to 30, inclusive) is selected. How many different lock combinations are possible assuming no number is repeated?

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$$_{30}p_{3} = \frac{\mathbf{30!}}{\mathbf{(30-3)!}} = \frac{30!}{27!} = \mathbf{30*29*28=24360}$$

Permutations

Practice:

From a club of 24 members, a President, Vice President, Secretary, Treasurer and Historian are to be elected. In how many ways can the offices be filled?

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$$ {}_{24}p_5 = \frac{\mathbf{24!}}{\mathbf{(24-5)!}} = \frac{24!}{19!} = $$

$$ \mathbf{24*23*22*21*20 = 5{,}100{,}480} $$

Combinations

A **Combination** is an arrangement of items in which order does not matter.

ORDER DOES NOT MATTER!

Since the order does not matter in combinations, there are fewer combinations than permutations. The combinations are a "subset" of the permutations.

Combinations

To find the number of Combinations of n items chosen r at a time, you can use the formula

$${}_nC_r = \frac{n!}{r!(n-r)!} \quad \text{where } 0 \le r \le n.$$

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$${}_5C_3 = \frac{\mathbf{5!}}{\mathbf{3!(5-3)!}} = \frac{5!}{3!2!} =$$

$$\frac{\mathbf{5*4*3*2*1}}{\mathbf{3*2*1*2*1}} = \frac{\mathbf{5*4}}{\mathbf{2*1}} = \frac{\mathbf{20}}{\mathbf{2}} = \mathbf{10}$$

Combinations

Practice:

To play a particular card game, each player is dealt five cards from a standard deck of 52 cards. How many different hands are possible?

Combinations

Practice: To play a particular card game, each player is dealt five cards from a standard deck of 52 cards. How many different hands are possible?

$${}_{52}C_5 = \frac{\mathbf{52!}}{\mathbf{5!(52-5)!}} = \frac{52!}{5!47!} =$$

$$\frac{\mathbf{52*51*50*49*48}}{\mathbf{5*4*3*2*1}} = \mathbf{2{,}598{,}960}$$

Combinations

Practice:

A student must answer 3 out of 5 essay questions on a test. In how many different ways can the student select the questions?

Combinations

Practice: A student must answer 3 out of 5 essay questions on a test. In how many different ways can the student select the questions?

$${}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5*4}{2*1} = 10$$

Combinations

Practice:

A basketball team consists of two centers, five forwards, and four guards. In how many ways can the coach select a starting line up of one center, two forwards, and two guards?

Combinations

Practice: A basketball team consists of two centers, five forwards, and four guards. In how many ways can the coach select a starting line up of one center, two forwards, and two guards?

Center: $${}_2C_1 = \frac{2!}{1!1!} = 2$$

Forwards: $${}_5C_2 = \frac{5!}{2!3!} = \frac{5*4}{2*1} = 10$$

Guards: $${}_4C_2 = \frac{4!}{2!2!} = \frac{4*3}{2*1} = 6$$

$${}_2C_1 * {}_5C_2 * {}_4C_2$$

Thus, the number of ways to select the starting line up is 2*10*6 = 120.

Guidelines on Which Method to Use

Permutations	Combinations
Number of ways of selecting r items out of n items	
Repetitions are not allowed	
Order is important.	Order is not important.
Arrangements of n items taken r at a time	Subsets of n items taken r at a time
${}_nP_r = n!/(n-r)!$	${}_nC_r = n!/[\,r!(n-r)!]$
Clue words: arrangement, schedule, order	Clue words: group, sample, selection

Example #1

- Two cards are picked without replacement from a deck of 52 playing cards. Determine the probability that both are kings using:
- (a) the multiplication law
- (b) combinations

Solution #1

Two cards are picked without replacement from a deck of 52 playing cards.
Determine the probability that both are kings using

a) the multiplication law

$P(K_1 \text{ and } K_2) = P(K_1)\cdot P(K_2|K_1)$

$= \left(\frac{4}{52}\right)\left(\frac{3}{51}\right) = \frac{1}{221}$

b) combinations

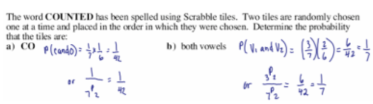
$P(K_1 \text{ and } K_2) = \frac{\text{favourable outcomes}}{\text{total possible outcomes}} = \frac{{}_4C_2}{{}_{52}C_2} = \frac{6}{1326} = \frac{1}{221}$

Example #2

- The word COUNTED has been spelled using Scrabble tiles. Two tiles are randomly chosen one at a time and placed in the order in which they were chosen. Determine the probability that the two tiles were:
- (a) CO
- (b) Both vowels

Again, use both the multiplication law and combinatorics to work out an answer.

Solution #2


The word **COUNTED** has been spelled using Scrabble tiles. Two tiles are randomly chosen one at a time and placed in the order in which they were chosen. Determine the probability that the tiles are:

a) **CO** $P(C \text{ and } O) = \frac{1}{7} \times \frac{1}{6} = \frac{1}{42}$

or $\frac{1}{{}_7P_2} = \frac{1}{42}$

b) both vowels $P(V_1 \text{ and } V_2) = \left(\frac{3}{7}\right)\left(\frac{2}{6}\right) = \frac{6}{42} = \frac{1}{7}$

or $\frac{{}_3P_2}{{}_7P_2} = \frac{6}{42} = \frac{1}{7}$

Example #3

- An athletic council decides to form a subcommittee of seven council members from the current council of 9 males and 6 females. What is the probability that the subcommittee will consist of exactly three females?

Solution #3

The Athletic Council decides to form a sub-committee of seven council members to look at how funds raised should be spent on sports activities at the school. There are a total of 15 athletic council members, 9 males and 6 females. What is the probability that the sub-committee will consist of exactly 3 females?

$$P(3F) = \frac{\text{\# ways to choose exactly 3 females}}{\text{\# ways to choose a subcommittee}} = \frac{{}_6C_3 \cdot {}_9C_4}{{}_{15}C_7} = \frac{56}{143}$$

Example #4

- A bag of marbles contains 5 red, 3 green and 6 blue marbles. If a child takes 3 marbles from the bag, determine the probability that:
- (a) exactly 2 are blue
- (b) at least one is blue
- (c) the first is red, the second green and the third is blue
- (d) one is red, one is green and one is blue

Solution #4

A bag of marbles contains 5 red, 3 green, and 6 blue marbles. If a child grabs three marbles from the bag, determine the probability that:

a) exactly 2 are blue

$$P(2B) = P(2B \text{ and } 1\text{other})$$

$$\frac{{}_6C_2 \cdot {}_8C_1}{{}_{14}C_3} = \frac{30}{91}$$

b) at least one is blue

$$P(1B : 2\text{other}) \text{ or } P(2B : 1\text{other}) \text{ or } P(3B : 0\text{others})$$

$$\frac{{}_6C_1 \cdot {}_8C_2 + {}_6C_2 \cdot {}_8C_1 + {}_6C_3 \cdot {}_8C_0}{{}_{14}C_3} = \frac{11}{13}$$

c) the first is red, the second is green and the third is blue — Order Important

$$P(1R \text{ and } 1G \text{ and } 1B)$$

$$\left(\frac{5}{14}\right)\left(\frac{3}{13}\right)\left(\frac{6}{12}\right) = \frac{15}{364}$$

$$\text{or } = \frac{{}_5P_1 \cdot {}_3P_1 \cdot {}_6P_1}{{}_{14}P_3} = \frac{15}{364} \checkmark$$

d) one is red, one is green and one is blue — Order Not Important

$$P(1R \text{ and } 1G \text{ and } 1B)$$

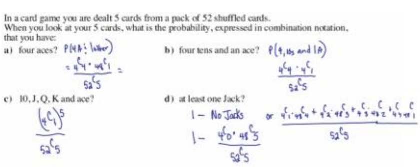
$$= 3! \times \left(\frac{5}{14}\right)\left(\frac{3}{13}\right)\left(\frac{6}{12}\right) = \frac{45}{182}$$

$$= \frac{{}_5C_1 \cdot {}_3C_1 \cdot {}_6C_1}{{}_{14}C_3} = \frac{45}{182} \checkmark$$

Example #5

- In a card game you are dealt 5 cards from a pack of 42 cards. When you look at your five cards, what is the probability (expressed in combinatoric form), that you have:
- (a) four aces
- (b) four tens and an ace
- (c) 10,J,Q,K,A
- (d) at least one Jack

Solution #5


In a card game you are dealt 5 cards from a pack of 52 shuffled cards.
When you look at your 5 cards, what is the probability, expressed in combination notation, that you have:

a) four aces? P(4A; 1 other) $= \frac{{}_4C_4 \cdot {}_{48}C_1}{{}_{52}C_5} =$

b) four tens and an ace? P(4 10s and 1A) $\frac{{}_4C_4 \cdot {}_4C_1}{{}_{52}C_5}$

c) 10, J, Q, K and ace? $\frac{({}_4C_1)^5}{{}_{52}C_5}$

d) at least one Jack? 1 − No Jacks

$1 - \frac{{}_4C_0 \cdot {}_{48}C_5}{{}_{52}C_5}$ or $\frac{{}_4C_1 \cdot {}_{48}C_4 + {}_4C_2 \cdot {}_{48}C_3 + {}_4C_3 \cdot {}_{48}C_2 + {}_4C_4 \cdot {}_{48}C_1}{{}_{52}C_5}$

Example 6

- In a class of thirty students, calculate the probability that :
- (a) they all have different birthdays
- (b) at least two of them have the same birthday

Solution #6

In a class of 30 students, calculate the probability (to the nearest hundredth) that:

a) they all have different birthdays (assume no one is born on February 29)

$$P(\text{all different Bdays}) = \frac{{}_{365}P_{30}}{365^{30}} = 0.29$$

b) at least 2 of them have the same birthday.

Compliment = 1 − No same Bdays

$1 - .29 = 0.71$

Example #7

- City council consists of 9 men and 6 women. Three representatives are chosen at random to form an environmental sub-committee.
- (a) What is the probability that Mayor Jim and two women are chosen?
- (b) what is the probability that two women are chosen if Mayor Jim must be on the committee?

Solution 7

City Council consists of nine men and six women. Three representatives are chosen at random to form a environmental sub-committee.

a) What is the probability that Mayor Jim Milonovich and two women are chosen?

$$P(\text{mayor and 2 women}) = \frac{{}_1C_1 \cdot {}_6C_2 \cdot {}_8C_0}{{}_{15}C_3} = \frac{15}{455} = \frac{3}{91}$$

b) What is the probability that two women are chosen if Mayor Jim Milonovich must be on the committee?

Mayor is already chosen

$$P(\text{No men and 2 women}) = \frac{{}_8C_0 \cdot {}_6C_2}{{}_{14}C_2} = \frac{15}{91}$$

Example

1) Beth and Shayna each purchase one raffle ticket. If a total of eleven raffle tickets are sold and two winners will be selected, what is the probability that both Beth and Shayna win?

Example

2) A meeting takes place between a diplomat and fourteen government officials. However, four of the officials are actually spies. If the diplomat gives secret information to ten of the attendees at random, what is the probability that no secret information was given to the spies?

Example

3) A fair coin is flipped ten times. What is the probability of the coin landing heads up exactly six times?

Example

4) A six-sided die is rolled six times. What is the probability that the die will show an even number exactly two times?

Example

5) A test consists of nine true/false questions. A student who forgot to study guesses randomly on every question. What is the probability that the student answers at least two questions correctly?

Example

6) A basketball player has a 50% chance of making each free throw. What is the probability that the player makes at least eleven out of twelve free throws?