Lesson 83

Equations of Planes

PLANES

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe.

- A single vector parallel to a plane is not enough to convey the ‘direction’ of the plane.

Image #1

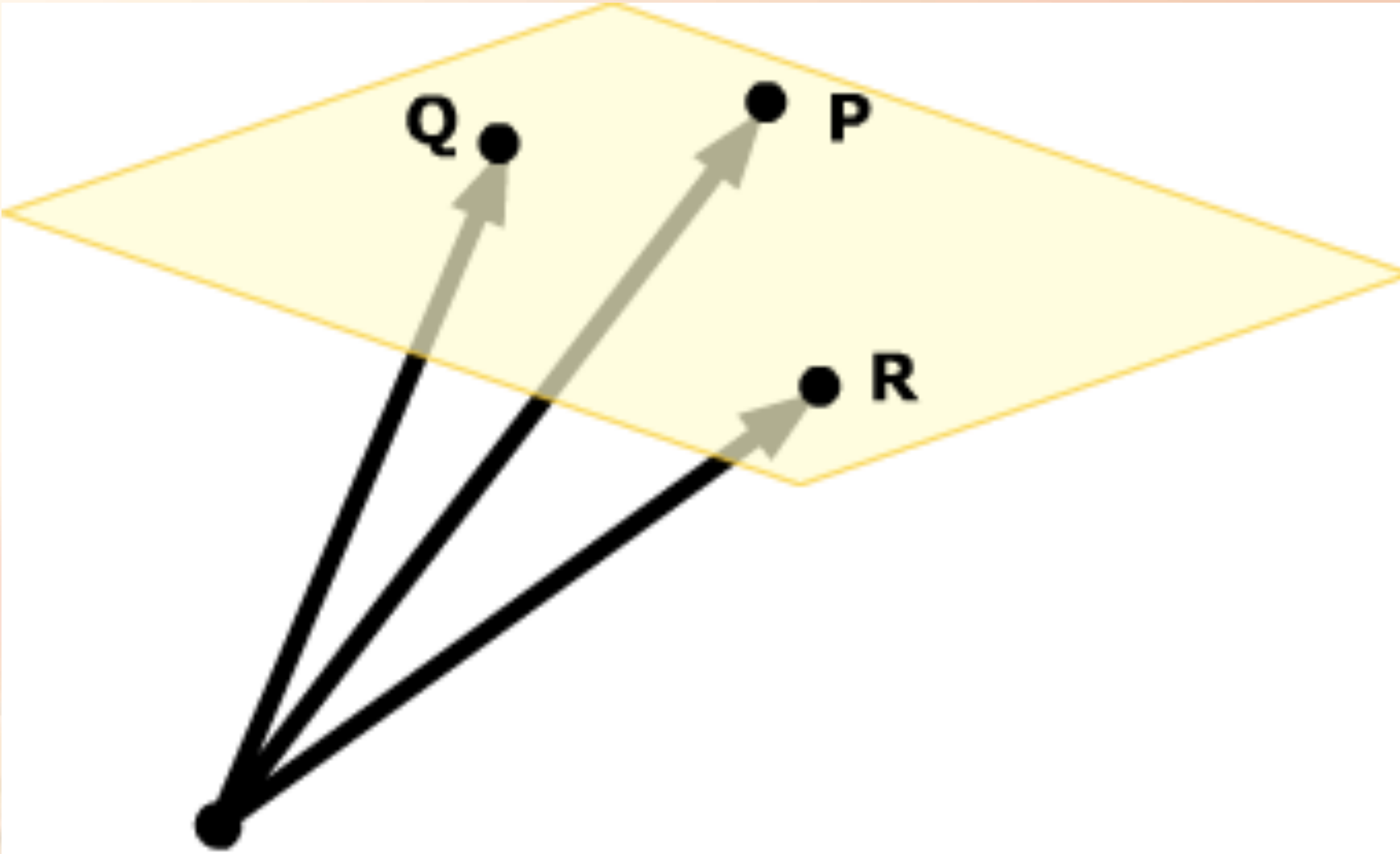
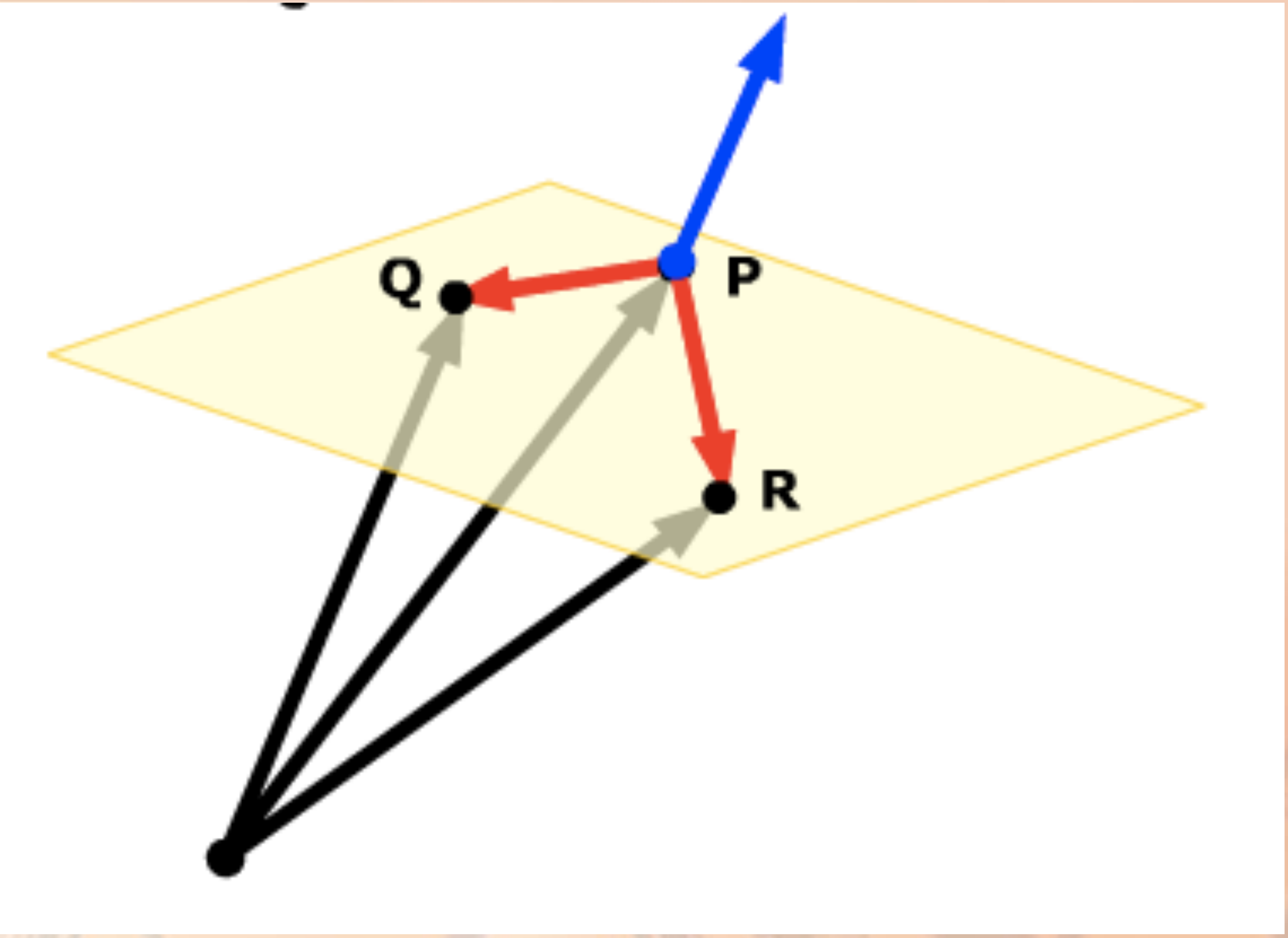


Image #2



Vector and Parametric Equations of Planes

The vector equation of a plane has the form:

$$\vec{r} = \vec{r}_0 + t\vec{a} + \lambda\vec{b}$$

Where

$\vec{a}$, $\vec{b}$ $\Rightarrow$ direction vectors in the plane

$\vec{r}_0$ $\Rightarrow$ position vector of a particular point in the plane

λ, t $\Rightarrow$ parameters

Example 1

- Determine the vector and parametric equations of the plane that contains the points A(1,0,-3), B(2,-3,1) and C(3,5,-3)

Example 2

- Does the point (4,5,-3) lie in the plane

$$\vec{r} = (4,1,6) + t(3,-2,1) + \lambda(-6,6,-1)$$

Example 3

- Find the vector equation of the plane that contains the two parallel lines

$$L_1 : \vec{r} = (2,4,1) + t(3,-1,1)$$
$$L_2 : \vec{r} = (1,4,4) + k(-6,2,-2)$$

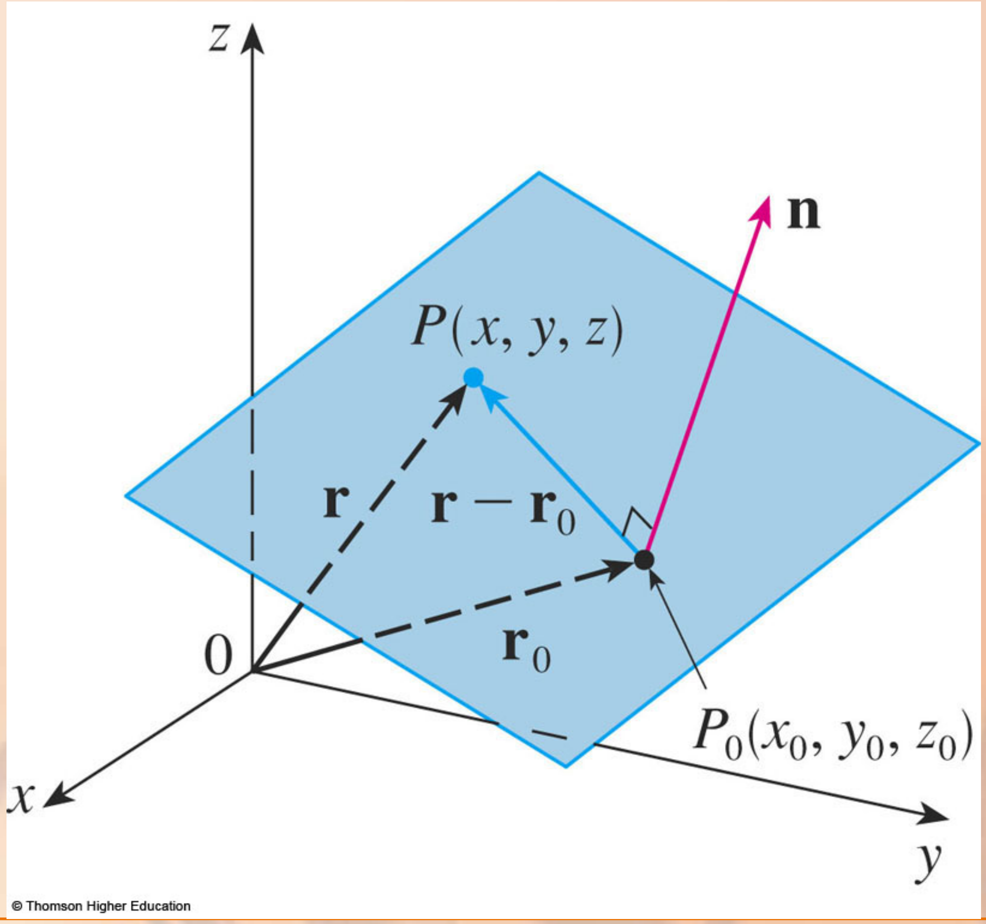
PLANES

However, a vector perpendicular to the plane does completely specify its direction.

Thus, a plane in space is determined by:

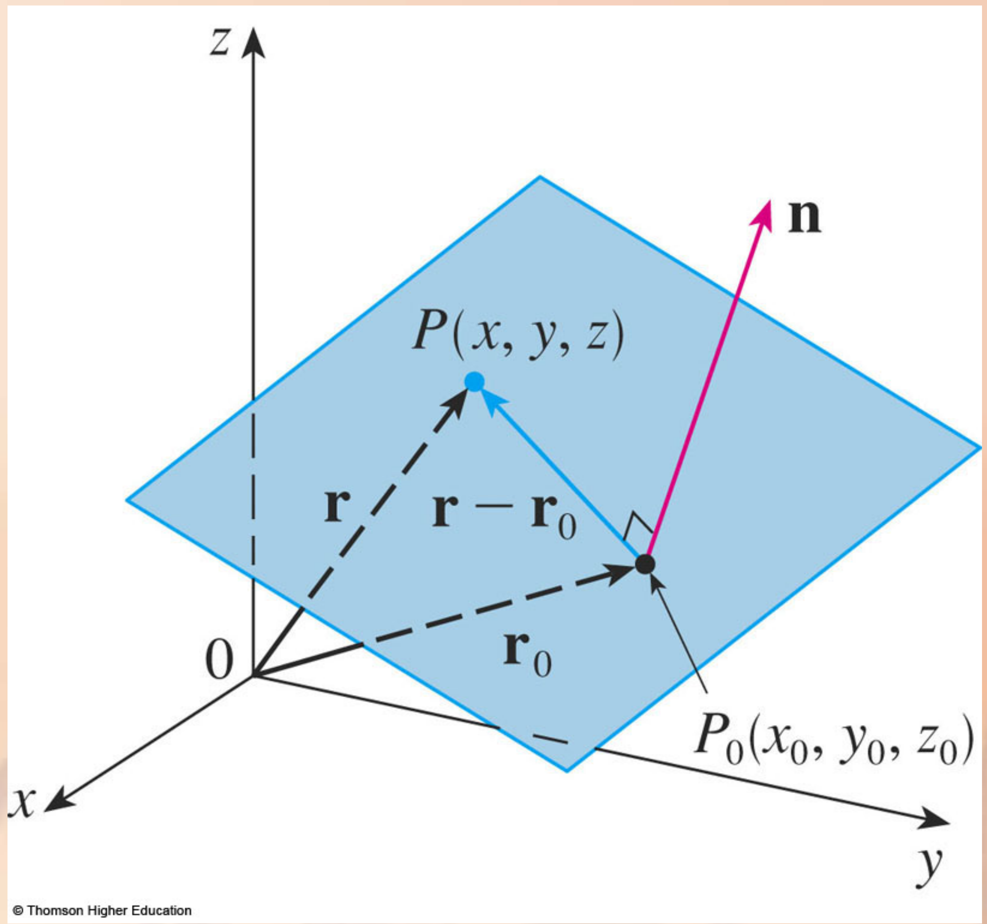
(1) A point $P_0(x_0, y_0, z_0)$ in the plane

(2) A vector **n** that is orthogonal to the plane



NORMAL VECTOR

This orthogonal vector **n** is called a normal vector.

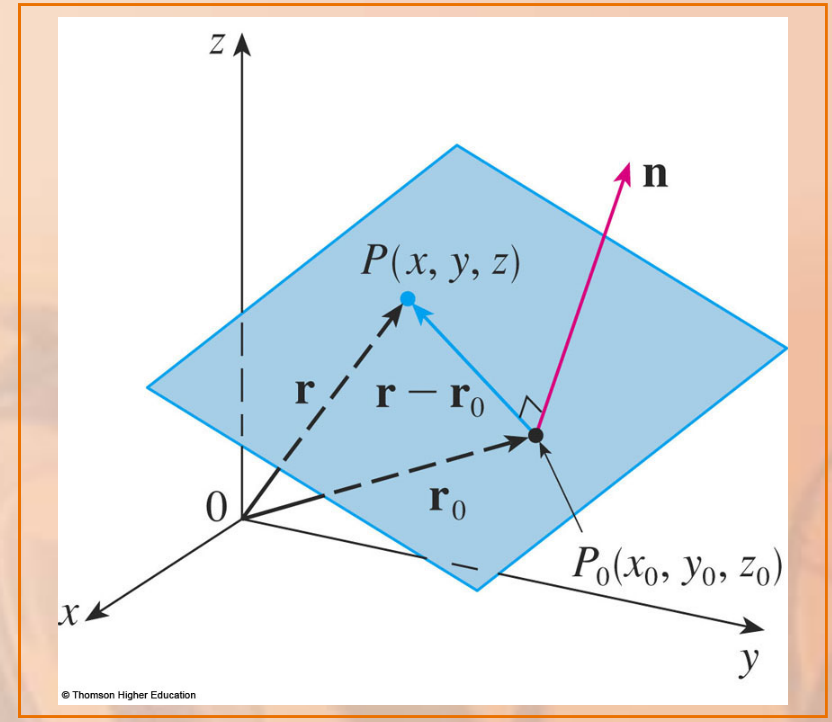


PLANES

Let $P(x, y, z)$ be an arbitrary point in the plane.

Let $\mathbf{r}_0$ and $\mathbf{r}_1$ be the position vectors of P_0 and P.

- Then, the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$



PLANES

The normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane.

In particular, $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r}_0$.

Thus, we have: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$

That can also be written as: $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$

EQUATIONS OF PLANES

To obtain a scalar equation for the plane, we write:

$$\mathbf{n} = \langle a, b, c \rangle$$

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$$

Then, the equation becomes:

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

That can also be written as:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

SCALAR EQUATION

That can also be written as:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- This equation is the scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector **n** = <*a, b, c*>.

Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$.

Find the intercepts and sketch the plane.

In Equation 7, putting

$a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, $z_0 = -1$,

we see that an equation of the plane is:

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

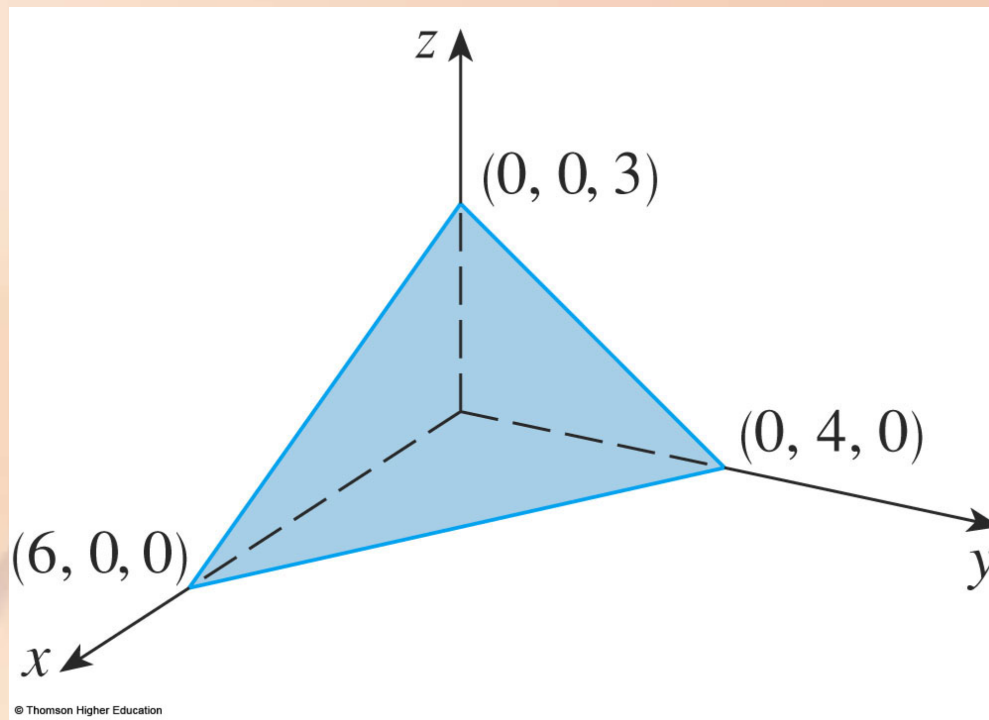
$$2x + 3y + 4z = 12$$

To find the x-intercept, we set $y = z = 0$ in the equation, and obtain $x = 6$.

Similarly, the y-intercept is 4 and the z-intercept is 3.

EQUATIONS OF PLANES **Example 1**

This enables us to sketch the portion of the plane that lies in the first octant.



EQUATIONS OF PLANES

By collecting terms in our equation as we did in our previous example, we can rewrite the equation of a plane as follows: $ax + by + cz + d = 0$

where $d = -(ax_0 + by_0 + cz_0)$

- This is called a linear equation in x, y, and z.

LINEAR EQUATION

$$ax + by + cz + d = 0$$

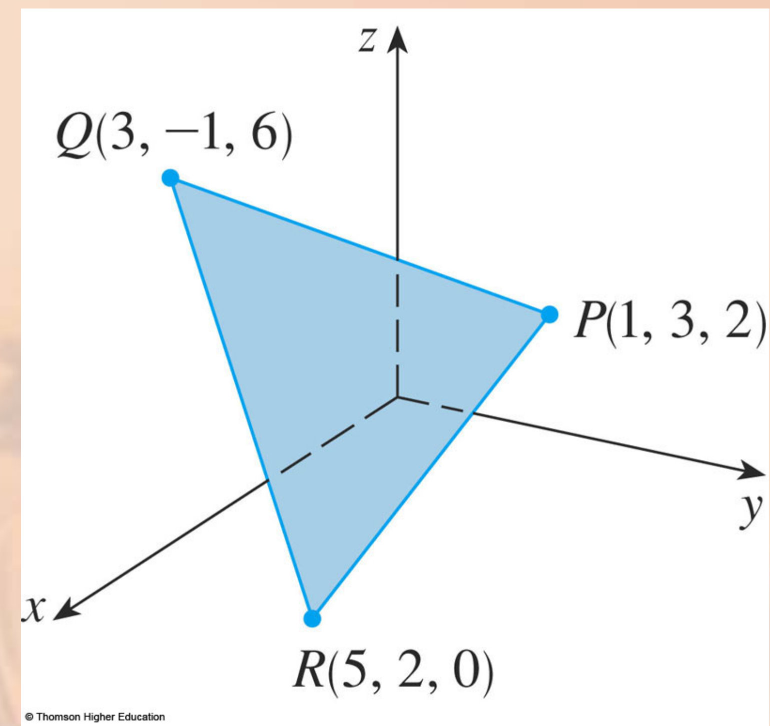
where $d = -(ax_0 + by_0 + cz_0)$

- This is called a linear equation in x, y, and z.

Example 2

Find a Cartesian equation of the plane that passes through the points

$P(1, 3, 2)$, $Q(3, -1, 6)$, $R(5, 2, 0)$



The vectors **a** and **b** corresponding to $\overline{PQ}$ and $\overline{PR}$ are:

a = <2, –4, 4> **b** = <4, –1, –2>

Since both **a** and **b** lie in the plane, their cross product **a** x **b** is orthogonal to the plane and can be taken as the normal vector.

Thus,

$$\mathbf{n} = \mathbf{a} \times \mathbf{b}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= 12\,\mathbf{i} + 20\,\mathbf{j} + 14\,\mathbf{k}$$

With the point $P(1, 2, 3)$ and the normal vector **n**, an equation of the plane is:

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

Find the point at which the line with parametric equations

$$x = 2 + 3t \qquad y = -4t \qquad z = 5 + t$$

intersects the plane

$$4x + 5y - 2z = 18$$

We substitute the expressions for x, y, and z from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

That simplifies to $-10t = 20$.

Hence, $t = -2$.

- Therefore, the point of intersection occurs when the parameter value is $t = -2$.

Then,

$$x = 2 + 3(-2) = -4$$

$$y = -4(-2) = 8$$

$$z = 5 - 2 = 3$$

- So, the point of intersection is (–4, 8, 3).

Example 4:

Given the normal vector, <3, 1, -2> to the plane containing the point (2, 3, -1), write the equation of the plane in both standard form and general form.

Solution:

Example 4:

Given the normal vector, <3, 1, -2> to the plane containing the point (2, 3, -1), write the equation of the plane in both standard form and general form.

Solution:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$3(x - 2) + 1(y - 3) - 2(z + 1) = 0$$

$$3x - 6 + y - 3 - 2z - 2 = 0$$

or

$$3x + y - 2z - 11 = 0$$

Example 5:

Given the points (1, 2, -1), (4, 0,3) and (2, -1, 5) in a plane, find the equation of the plane in general form.

Example 5: Given the points (1, 2, -1), (4, 0,3) and (2, -1, 5) in a plane, find the equation of the plane in general form.

Solution: To write the equation of the plane we need a point (we have three) and a vector normal to the plane. So we need to find a vector normal to the plane. First find two vectors in the plane, then recall that their cross product will be a vector normal to both those vectors and thus normal to the plane.

Two vectors: From (1, 2, -1) to (4, 0, 3): < 4-1, 0-2, 3+1 > = <3,-2,4>

From (1, 2, -1) to (2, -1, 5): < 2-1, -1-2, 5+1 > = <1,-3,6>

Their cross product:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 4 \\ 1 & -3 & 6 \end{vmatrix} = 0\vec{i} - 14\vec{j} - 7\vec{k} = -14\vec{j} - 7\vec{k}$$

Equation of the plane:

$$0(x-1) - 14(y-2) - 7(z+1) = 0$$

$$-14y - 7z + 21 = 0$$

or

$$2y + z - 3 = 0$$

Example 6.

Show that $\underline{\boldsymbol{n}} = \begin{bmatrix} \mathbf{3} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ is perpendicular to the plane

containing the points $A(1,\ 0,\ 2)$, $B(2,\ 3,\ -1)$ and $C(2,\ 2,\ -1\ )$.

6. Show that $\underline{\boldsymbol{n}} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ is perpendicular to the plane containing the points $A(1, 0, 2)$, $B(2, 3, -1)$ and $C(2, 2, -1)$.

Solution:

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \quad \overrightarrow{AC} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\underline{\boldsymbol{n}}.\overrightarrow{AB} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} = 0, \quad \underline{\boldsymbol{n}}.\overrightarrow{AC} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = 0$$

$\underline{n}$ is perpendicular to 2 vectors in the plane so is perpendicular to the plane.