L82 VECTOR EQUATIONS OF LINES

HL Math - Santowski

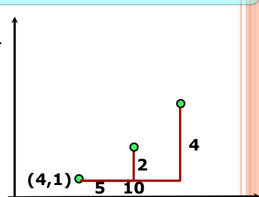
Vector equation of a line 1

A plane starts a journey at the point (4,1) and moves each hour along the vector $\begin{pmatrix}5\\2\end{pmatrix}$.

a) Find the plane's coordinate after 1 hour.

b) Find the plane's coordinate after 2 hours.

c) Find the plane's coordinate after *t* hours.



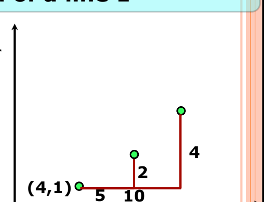
Vector equation of a line 1

A plane starts a journey at the point (4,1) and moves each hour along the vector $\begin{pmatrix}5\\2\end{pmatrix}$.

a) Find the plane's coordinate after 1 hour. **(9,3)**

b) Find the plane's coordinate after 2 hours. **(14,5)**

c) Find the plane's coordinate after *t* hours.



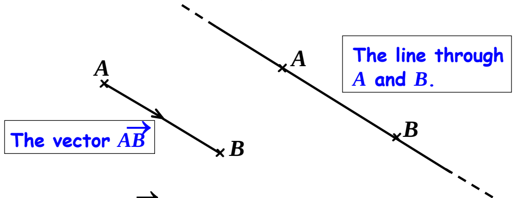
Vector equation $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}+t\begin{pmatrix}5\\2\end{pmatrix}$

A coordinate on the line.

The vector part of the line.

Parametric equation $x=4+5t,\ y=1+2t$

The vector $\overrightarrow{AB}$ and the vector equation of the line AB are very different things.



The vector $\overrightarrow{AB}$ has a definite length (magnitude).

The line AB is a line passing through the points A and B and has infinite length.

Finding the Equation of a Line

In coordinate geometry, the equation of a line is

$$y = mx + b$$

e.g. $y = -2x + 3$

The equation gives the value (coordinate) of y for any point which lies on the line.

The vector equation of a line must give us the position vector of any point on the line.

We start with fixing a line in space.

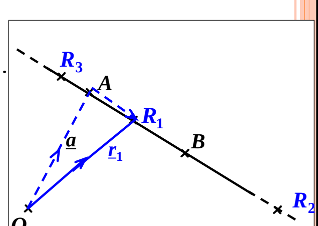
We can do this by fixing 2 points, A and B. There is only one line passing through these points.

A and B are fixed points.

We consider several more points on the line.

We need an equation for $\underline{r}$, the position vector of any point R on the line.

Starting with R_1:



$$\underline{r}_1 = \underline{a} + \tfrac{1}{2}\overrightarrow{AB}$$

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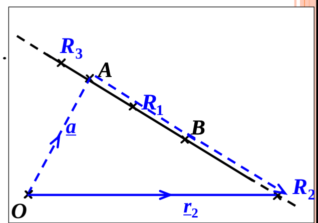

$$\underline{r}_1 = \underline{a} + \tfrac{1}{2}\overrightarrow{AB}$$

$$\underline{r}_2 = \underline{a} + 2\overrightarrow{AB}$$

A and B are fixed points.

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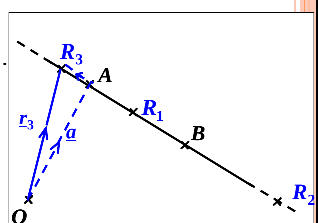
Starting with R_1:



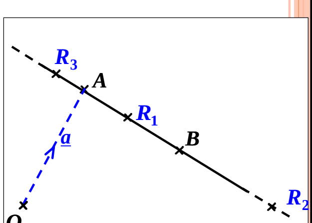
$$\underline{r}_1 = \underline{a} + \tfrac{1}{2}\overrightarrow{AB}$$

$$\underline{r}_2 = \underline{a} + 2\overrightarrow{AB}$$

$$\underline{r}_3 = \underline{a} + (-\tfrac{1}{4})\overrightarrow{AB}$$

So for R_1, R_2 and R_3

$$\boxed{\begin{aligned}\underline{r}_1 &= \underline{a} + \tfrac{1}{2}\overrightarrow{AB}\\ \underline{r}_2 &= \underline{a} + 2\overrightarrow{AB}\\ \underline{r}_3 &= \underline{a} + (-\tfrac{1}{4})\overrightarrow{AB}\end{aligned}}$$



For any position of R, we have

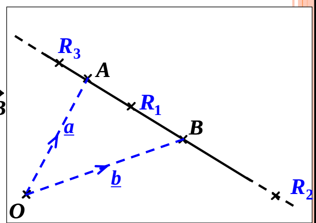
$$\boxed{\underline{r} = \underline{a} + t\,\overrightarrow{AB}}$$

t is called a parameter and can have any real value.
It is a scalar not a vector.

$$\underline{r} = \underline{a} + t\,\overrightarrow{AB}$$

We can substitute for $\overrightarrow{AB}$

$$\underline{r} = \underline{a} + t(\underline{b} - \underline{a})$$



Instead of using $\underline{a}$ here . . . we could use $\underline{b}$.

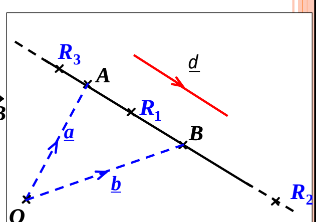
The value of t would then be different to get to any particular point.

We say the equation is not unique.

$$\underline{r} = \underline{a} + t\,\overrightarrow{AB}$$

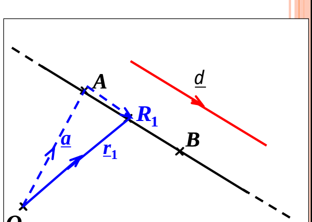
We can substitute for $\overrightarrow{AB}$

$$\underline{r} = \underline{a} + t(\underline{b} - \underline{a})$$



Also, instead of $\overrightarrow{AB}$ we can equally well use any vector $\underline{p}$ which is parallel to AB.

If $\underline{d}$ is not the same length as AB, t will have a different value for any particular R.



e.g. $\quad \underline{r}_1 = \underline{a} + \frac{1}{2}\overrightarrow{AB}$

or $\quad \underline{r}_1 = \underline{a} + \frac{1}{3}\underline{d}$

SUMMARY

- The vector equation of the line through 2 fixed points A and B is given by

$$\underline{r} = \underline{a} + t\,\overrightarrow{AB} \quad \Rightarrow \quad \underline{r} = \underline{a} + t(\underline{b} - \underline{a})$$

- The vector equation of the line through 1 fixed point A and parallel to the vector $\underline{d}$ is given by

$$\underline{r} = \underline{a} + t\underline{d}$$

position vector . . . of a known point on the line

direction vector . . . of the line

You may find the equation of the line through A and B written as $\underline{r} = \underline{a} + t\,\underline{b}$.

I am not going to do this as it doesn't emphasise the vital difference between the position vector of a point on the line and the direction vector of the line.

I will use $\underline{r} = \underline{a} + t\underline{d}$

where $\underline{d}$ is a ***direction*** vector on, or parallel to, the line.

I will, however, vary the letters for the parameter.

The most common parameters are s, t, λ and μ.

e.g. 1 Find the equation of the line passing through the points A and B with position vectors $\underline{a}$ and $\underline{b}$ where

$$\underline{a} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

Solution:

e.g. 1 Find the equation of the line passing through the points A and B with position vectors $\underline{a}$ and $\underline{b}$ where

$$\underline{a} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

Solution: $\boxed{\underline{r} = \underline{a} + t\underline{d}}$

$$\underline{d} = \underline{b} - \underline{a} \quad \Rightarrow \quad \underline{d} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ -1 \end{bmatrix}$$

So, $\underline{r} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -5 \\ 5 \\ -1 \end{bmatrix}$

In this example we had

$$\underline{a} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

giving $\underline{r} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -5 \\ 5 \\ -1 \end{bmatrix}$

We can replace $\underline{r}$ with $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and/or replace $\underline{a}$ with $\underline{b}$

e.g. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} + t\begin{bmatrix} -5 \\ 5 \\ -1 \end{bmatrix}$

So,

$\underline{r} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -5 \\ 5 \\ -1 \end{bmatrix}$ is the same line as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} + t\begin{bmatrix} -5 \\ 5 \\ -1 \end{bmatrix}$

However, the value of t for any particular point has now changed.

Which point is given by $t = 2$ in the 1st version?

ANS: $\begin{bmatrix} -8 \\ 9 \\ 0 \end{bmatrix}$

What value of t in the 2nd version gives the same point?

ANS: $t = 1$

e.g. 2 Find the equation of the line passing through the point $A(-1,3,4)$, parallel to the vector

$$\underline{b} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

Solution:

e.g. 2 Find the equation of the line passing through the point $A(-1,3,4)$, parallel to the vector

$$\underline{b} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

Solution: $\boxed{\underline{r} = \underline{a} + t\underline{d}}$

$\underline{d} = \underline{b}$

So, $\underline{r} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + t\begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + t\begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$

e.g. 3 Show that C with position vector $\underline{c} = \begin{bmatrix} 7 \\ -6 \\ 3 \end{bmatrix}$

lies on the line $\underline{r} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -5 \\ 5 \\ -1 \end{bmatrix}$

We have $\begin{bmatrix} 7 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -5 \\ 5 \\ -1 \end{bmatrix}$

The top row of the vectors gives the x-components, so,

$$7 = 2 + t(-5)$$
$$\Rightarrow \quad 5t = 2 - 7 \quad \Rightarrow \quad t = -1$$

However, for the point to lie on the line, this value of t must also give the y- and z- components.

y: $-1 + 5t$ $= -1 + 5(-1)$ $= -6$

z: $2 - t$ $= 2 - (-1)$ $= 3$

Since all 3 equations are satisfied, C lies on the line.

Exercise

1. Find a vector equation for the line AB for each of the following:
   (a) $A(2, -3, -1)$, $B(-2, 3, -3)$
   (b) $\underline{a} = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$
   (c) $\underline{a} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and AB is parallel to the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. Does the point $(0, 0, 2)$ lie on the line AB in 1(a)?

Solutions

(a) $A(2, -3, -1)$, $B(-2, 3, -3)$

$\boxed{\underline{r} = \underline{a} + t\,\underline{p}} \quad \Rightarrow \quad \underline{r} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} + t \begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}$

(b) $\underline{a} = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} \Rightarrow \underline{r} = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$

(c) $\underline{a} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ parallel to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \underline{r} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solutions

2. Does the point $(0, 0, 2)$ lie on the line AB in 1(a)?

From 1(a), $\underline{r} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} + t\begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}$

x: $0 = 2 - 4t \quad \Rightarrow \quad 4t = 2 \quad \Rightarrow \quad t = \frac{1}{2}$

y: $-3 + 6t$
$= -3 + 6\left(\frac{1}{2}\right)$
$= -3 + 3$
$= 0$

z: $-1 - 2t$
$= -1 - 2\left(\frac{1}{2}\right)$
$= -1 - 1$
$= -2 \quad \neq 2$

The point does not lie on AB.

The Cartesian Form of the Equation of a Line

The equation $y = mx + b$ is the Cartesian equation of a line but only if it lies in the x-y plane.

The more general form can be easily found from the vector form.

The Cartesian form does not contain a parameter.

e.g. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix} + t\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

We can extract the 3 components from this equation:

$x = -3 + 2t \qquad y = 1 + 3t \qquad z = -4 + 4t$

To eliminate the parameter, t, we rearrange to find t:

$\frac{x+3}{2} = t \qquad \frac{y-1}{3} = t \qquad \frac{z+4}{4} = t$

So, $\frac{x+3}{2} = \frac{y-1}{3} = \frac{z+4}{4} \; (= t)$

This 3 part equation is the Cartesian equation.

SUMMARY

The Cartesian equation of a line is given by

$$\frac{x-a_1}{p_1}=\frac{y-a_2}{p_2}=\frac{z-a_3}{p_3} \qquad p_1, p_2, p_3 \neq 0$$

where the position vector of the point on the line is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

and the direction vector is $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$

It is easy to make a mistake when writing the Cartesian equation of a line.

Also, it isn't obvious what to do if an element of $\underline{p}$ is zero.

For both these reasons you may prefer to rearrange the vector equation to find the parameter rather than quote the formula.

e.g. Find the Cartesian equation of the line

$$\underline{r} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

Solution:

$$x=-3 \quad ------- \ (1)$$
$$y=2-t \quad ------- \ (2)$$
$$z=1+2t \quad ------- \ (3)$$

(1) doesn't contain t so just gives $x=-3$

$(2) \Rightarrow \ t=\frac{y-2}{-1} \qquad (3) \Rightarrow \ t=\frac{z-1}{2}$

So the line is $\quad x=-3; \quad \frac{y-2}{-1}=\frac{z-1}{2}$

Exercise

1. Write the following lines in Cartesian form:

(a) $\underline{r} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} + t\begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}$ (b) $\underline{r} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

Answers: (a) $\dfrac{x-2}{-4} = \dfrac{y+3}{6} = \dfrac{z+1}{-2}$

(b) $\dfrac{x-1}{-2} = \dfrac{z+2}{3}; \quad y = 1$

Converting from Vector to Cartesian Form

If we are given the vector equation of a line and we want to write it in Cartesian form we can do this by writing the vector equation as a set of parametric equations. For example,

Find the Cartesian equation of the line **r** = 5**i** – **j** + t(-2**i** + 3**j**).

Since r is a general point on the line we can write it as $\begin{pmatrix} x \\ y \end{pmatrix}$.

The vector equation of the line can therefore be written as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

The equation of the line is therefore given by the parametric equations:

$x = 5 - 2t$
$y = -1 + 3t$

Converting from Vector to Cartesian Form

Rearranging to make t the subject of these equations gives:

$$t = \frac{5-x}{2}$$
$$t = \frac{y+1}{3}$$

Equating these gives us the Cartesian form of the equation of the line.

$$\frac{y+1}{3} = \frac{5-x}{2}$$
$$2y + 2 = 15 - 3x$$
$$2y + 3x - 13 = 0$$

The same method can be used for a line given in three dimensions.

Converting from vector to Cartesian form

Find the Cartesian equation of the line
r = 6**i** + 3**j** – 4**k** + t(5**i** – 4**j** + 7**k**).

Since r is a general point on the line we can write it as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The vector equation of the line can therefore be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -4 \end{pmatrix} + t\begin{pmatrix} 5 \\ -4 \\ 7 \end{pmatrix}$$

The equation of the line is therefore given by the parametric equations:

$x = 6 + 5t$

$y = 3 - 4t$

$z = -4 + 7t$

Converting from vector to Cartesian form

Rearranging to make t the subject of these equations gives:

$$t = \frac{x-6}{5}$$
$$t = \frac{3-y}{4}$$
$$t = \frac{z+4}{7}$$

Equating these gives us the Cartesian form of the equation of the line.

$$\frac{x-6}{5} = \frac{3-y}{4} = \frac{z+4}{7}$$

In general the Cartesian form of a line given by the vector equation **r** = a_1**i** + a_2**j** + a_3**k** + t(b_1**i** + b_2**j** + b_3**k**) is

$$\boxed{\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}}$$

Vector equation of a line 2

Find the vector and parametric equations of the straight line that passes through $A(x_1,y_1)$ and $B(x_2,y_2)$.

Find the vector that connects A to B.

$$AB = b - a = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

Vector equation will be**:**

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

Or using the B coordinate and the vector BA.

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + s\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$$

Parametric equation will be:

$x = x_1 + t(x_2 - x_1), y = y_1 + t(y_2 - y_1)$ **or**

$x = x_2 + s(x_1 - x_2), y = y_2 + s(y_1 - y_2)$

Find the vector and parametric equations of the straight line that passes through A and B that have position vectors $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

Vector equation:

$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ **or** $\begin{pmatrix} 6 \\ 2 \end{pmatrix} + s\begin{pmatrix} -5 \\ 1 \end{pmatrix}$

Parametric equation:

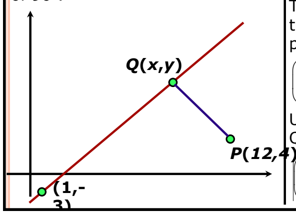
$x = 1 + 5t, y = 3 - t$ **or**

$x = 6 - 5s, y = 2 + s$

Shortest distance problems

Find the shortest distance between the point $P(12,4)$ and the straight line with the vector equation, $\begin{pmatrix}1\\-3\end{pmatrix}+t\begin{pmatrix}3\\5\end{pmatrix}$

The shortest distance from any point to a line will make an angle of 90°.



Any point on the line will have the coordinates:

$x = 1 + 3t, y = -3 + 5t$

(the parametric equation of the line)

Find the vector QP.

$QP = p - q = \begin{pmatrix}12\\4\end{pmatrix} - \begin{pmatrix}1+3t\\-3+5t\end{pmatrix} = \begin{pmatrix}11-3t\\7-5t\end{pmatrix}$

The vector QP and vector part of the line meet at 90°.So the dot product of the vectors will be 0**.**

$\begin{pmatrix}11-3t\\7-5t\end{pmatrix}.\begin{pmatrix}3\\5\end{pmatrix} = 0$

$33 - 9t + 35 - 25t = 0$

$34t = 68$

$t = 2$

Use this to find the magnitude of QP.

$\left|\begin{pmatrix}11-3t\\7-5t\end{pmatrix}\right| = \left|\begin{pmatrix}5\\-3\end{pmatrix}\right|$

$|QP| = \sqrt{34}$

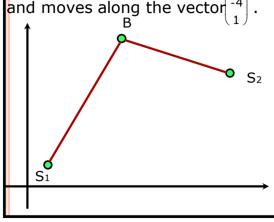
Intersecting lines

Vector lines can intersect, although they do not have to.

Example

2 ships plan to meet at a buoy (B). Ship 1 starts at (2,3) and moves along the vector $\begin{pmatrix}1\\3\end{pmatrix}$.

Ship 2 starts at (13,10) and moves along the vector $\begin{pmatrix}-4\\1\end{pmatrix}$.



Write down two vector line equations.

$S_1 = \begin{pmatrix}2\\3\end{pmatrix} + t\begin{pmatrix}1\\3\end{pmatrix} \quad S_2 = \begin{pmatrix}13\\10\end{pmatrix} + u\begin{pmatrix}-4\\1\end{pmatrix}$

B has coordinates (x,y).

$(x =)2 + t = 13 - 4u$

$(y =)3 + 3t = 10 + u$

Solving this gives t=3, u=2.

Check this out with both lines gives the coordinates:

B (5,12)

Questions

1. Find the shortest distance from the point P(20,3) to the straight line with parametric equations,

$x = 1 + 4t, y = -5 + 3t$

Make Q be the point on the line where QP and the line meet at 90°.

Find the vector QP.

Find the value of t using the dot product.

Find the numerical coordinates of Q.

Find the magnitude of QP.

***QP*=5**

2. Two ants set off to meet each other at point M. The first ant starts at (7,1) and the second ant starts at (18,13). The ants are moving along the vectors,

$A_1 = \begin{pmatrix}1\\2\end{pmatrix}, A_2 = \begin{pmatrix}-3\\-1\end{pmatrix}$

a) Find the coordinates of M.

M (12,11)

b) Find the distance that the first ant covers.

$5\sqrt{5}$

c) Find the distance that the second ant covers.

$2\sqrt{10}$