Lesson 81: The Cross Product of Vectors

IBHL - SANTOWSKI

In this lesson you will learn
- how to find the cross product of two vectors
- how to find an orthogonal vector to a plane defined by two vectors
- how to find the area of a parallelogram given two vectors
- how to find the volume of a parallelepiped given three vectors

CROSS PRODUCT IN PHYSICS

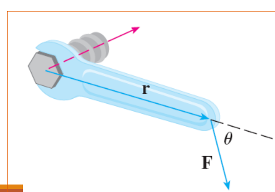
The idea of a cross product occurs often in physics.

CROSS PRODUCT IN PHYSICS

In particular, we consider a force **F** acting on a rigid body at a point given by a position vector **r**.

CROSS PRODUCT IN PHYSICS

For instance, if we tighten a bolt by applying a force to a wrench, we produce a turning effect.

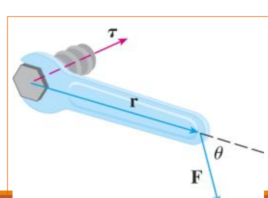


TORQUE

The torque τ (relative to the origin) is defined to be the cross product of the position and force vectors

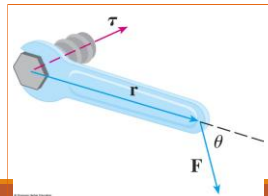
$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

- It measures the tendency of the body to rotate about the origin.



TORQUE

The direction of the torque vector indicates the axis of rotation.



TORQUE

The magnitude of the torque vector is

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \vartheta$$

where ϑ is the angle between the position and force vectors.

TORQUE

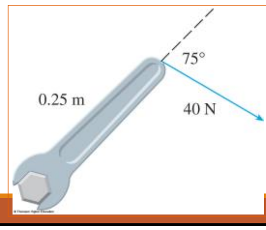
Observe that the only component of **F** that can cause a rotation is the one perpendicular to **r**—that is, $|\mathbf{F}| \sin \vartheta$.

- The magnitude of the torque is equal to the area of the parallelogram determined by **r** and **F**.

TORQUE

A bolt is tightened by applying a 40-N force to a 0.25-m wrench, as shown.

- Find the magnitude of the torque about the center of the bolt.



TORQUE

The magnitude of the torque vector is:

$$|\tau| = |\mathbf{r} \times \mathbf{F}|$$
$$= |\mathbf{r}||\mathbf{F}| \sin 75^\circ$$
$$= (0.25)(40) \sin 75^\circ$$
$$= 10 \sin 75^\circ$$
$$\approx 9.66 \text{ N}\cdot\text{m}$$

TORQUE

If the bolt is right-threaded, then the torque vector itself is

$$\tau = |\tau|\ \mathbf{n} \approx 9.66\ \mathbf{n}$$

where **n** is a unit vector directed down into the slide.

Objective 1

FINDING THE CROSS PRODUCT OF TWO VECTORS

The Cross Product
The cross product of two vectors, denoted as $\vec{a} \times \vec{b}$, unlike the dot product, represents a vector.

The cross product is defined to be for

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \ \ and \ \ \vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

You are probably wondering if there is an easy way to remember this.

The easy way is to use determinants of size 3 x 3.

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \ \ and \ \ \vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Let's set up a 3 x 3 determinant as follows:

1. First use the unit vectors i , j, and k as the first row of the determinant.
2. Use row 2 for the components of **a** and row 3 for the components of **b**. We will expand by the first row using minors.

$$\begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{\mathrm{i}}\left(a_2 b_3 - a_3 b_2\right) - \vec{\mathrm{j}}\left(a_1 b_3 - a_3 b_1\right) + \vec{\mathrm{k}}\left(a_1 b_2 - a_2 b_1\right) =$$

$$\left(a_2 b_3 - a_3 b_2\right)\vec{\mathrm{i}} + \left(a_3 b_1 - a_1 b_3\right)\vec{\mathrm{j}} + \left(a_1 b_2 - a_2 b_1\right)\vec{\mathrm{k}}$$

Don't forget to change the sign to – for the j expansion.

Find the cross product for the vectors below.

$$\vec{a} = \langle 2,4,5 \rangle \;\; and \;\; \vec{b} = \langle 1,-2,-1 \rangle$$

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$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & 5 \\ 1 & -2 & -1 \end{vmatrix} =$$

$$(-4+10)\vec{i} - (-2-5)\text{j} + (-4-4)\text{k} = 6\text{i} + 7\text{j} - 8\text{k}$$

CROSS PRODUCT EXAMPLE

If **a** = ⟨1, 3, 4⟩ and **b** = ⟨2, 7, –5⟩, then

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If **a** = ⟨1, 3, 4⟩ and **b** = ⟨2, 7, –5⟩, then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}\mathbf{k}$$

$$= (-15-28)\mathbf{i} - (-5-8)\mathbf{j} + (7-6)\mathbf{k}$$

$$= -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

CROSS PRODUCT

Show that **a** x **a** = **0** for any vector **a** in V_3.

- If **a** = ⟨a_1, a_2, a_3⟩, then

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- If **a** = ⟨a_1, a_2, a_3⟩, then

$$\mathbf{a}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= (a_2a_3 - a_3a_2)\mathbf{i} - (a_1a_3 - a_3a_1)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k}$$

$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

Since the cross product is determined by using determinants, we can understand the algebraic properties of the Cross Product which are:

- $\vec{u} \times \vec{v} = -\left(\vec{v} \times \vec{u}\right)$ Which would come from the fact that if you interchange two rows of a determinant you negate the determinant.
- $\vec{u} \times \left(\vec{v} + \vec{w}\right) = \left(\vec{u} \times \vec{v}\right) + \left(\vec{u} \times \vec{w}\right)$
- $c\left(\vec{u} \times \vec{v}\right) = \left(c\vec{u}\right) \times \vec{v} = \vec{u} \times \left(c\vec{v}\right)$
- $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
- $\vec{u} \times \vec{u} = \vec{0}$
- $\vec{u} \bullet \left(\vec{v} \times \vec{w}\right) = \left(\vec{u} \times \vec{v}\right) \bullet \vec{w}$

Objective 2

FIND AN ORTHOGONAL VECTOR TO A PLANE DEFINED BY TWO VECTORS

Now that you can do a cross product the next step is to see why this is useful.

Let's look at the 3 vectors from the last problem

$\vec{a} = \langle 2,4,5 \rangle$, $\vec{b} = \langle 1,-2,-1 \rangle$ *and* $\vec{a} \times \vec{b} = \langle 6,7,-8 \rangle$

What is the dot product of

$\vec{a} = \langle 2,4,5 \rangle$ *with* $\vec{a} \times \vec{b} = \langle 6,7,-8 \rangle$

And $\vec{b} = \langle 1,-2,-1 \rangle$ *with* $\vec{a} \times \vec{b} = \langle 6,7,-8 \rangle$?

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If you answered 0 in both cases, you would be correct. Recall that whenever two non-zero vectors are perpendicular, their dot product is 0. Thus the cross product creates a vector perpendicular to the vectors a and b. Orthogonal is another name for perpendicular.

Show that $\vec{u} \bullet (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \bullet \vec{w}$ is true for
$\vec{u} = \langle 1,2,3 \rangle, \vec{v} = \langle -1,-1,0 \rangle, \text{and } \vec{w} = \langle 5,0,-1 \rangle$

Solution:

Show that $\vec{u} \bullet (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \bullet \vec{w}$ is true for
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Solution: $\vec{u} = \langle 1,2,3 \rangle, \vec{v} = \langle -1,-1,0 \rangle, \text{and } \vec{w} = \langle 5,0,-1 \rangle$

The left hand side

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 0 \\ 5 & 0 & -1 \end{vmatrix} = \vec{i} - \vec{j} + 5\vec{k}$$

$\langle 1,2,3 \rangle \bullet \langle 1,-1,5 \rangle = 1 - 2 + 15 = 14$

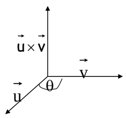
The right hand side

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -1 & -1 & 0 \end{vmatrix} = 3\vec{i} - 3\vec{j} + \vec{k}$$

$\langle 3,-3,1 \rangle \bullet \langle 5,0,-1 \rangle = 15 + 0 - 1 = 14$

Geometric Properties of the Cross Product

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors and let θ (the Greek letter theta) be the angle between $\vec{u}$ and $\vec{v}$

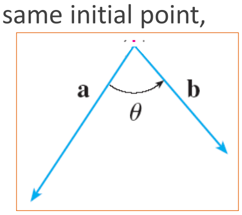


1. $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

Remember: orthogonal means perpendicular to

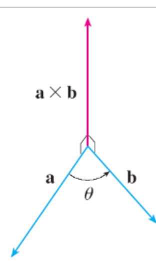
CROSS PRODUCT

Let **a** and **b** be represented by directed line segments with the same initial point, as shown.



CROSS PRODUCT

The cross product **a** x **b** points in a direction perpendicular to the plane through **a** and **b**.

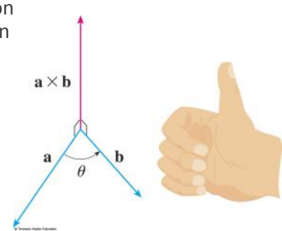


CROSS PRODUCT

It turns out that the direction of **a** x **b** is given by the right-hand rule, as follows.

RIGHT-HAND RULE

If the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from **a** to **b**, then your thumb points in the direction of **a** x **b**.



Geometric Properties of the Cross Product

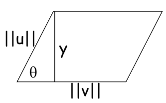
We know the direction of the vector **a** x **b**.

The remaining thing we need to complete its geometric description is its length |**a** x **b**|.

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors and let θ be the angle between $\vec{u}$ and $\vec{v}$, then $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$

3. $\vec{u} \times \vec{v} = \vec{0}$ *if and only if* $\vec{u}$ *and* $\vec{v}$ *are multiples of each other*

4. $\|\vec{u} \times \vec{v}\| =$ *area of the paralleogram having* $\vec{u}$ *and* $\vec{v}$ *as adjacent sides.*



Proof: The area of a parallelogram is base times height. A = bh

sin θ = y/||u||

||u||sin θ = y = height

||v||•y = ||v|| ||u||sin θ = $\|\vec{u} \times \vec{v}\|$

Example problem for property 1

Find 2 unit vectors perpendicular to **a** = 2 **i** - **j** + 3**k** and **b** = -4**i** + 2**j** - **k**.

Solution

.

Example problem for property 1

Find 2 unit vectors perpendicular to **a** = 2 **i** - **j** + 3**k** and **b** = -4**i** + 2**j** - **k**.

Solution

The two given vectors define a plane.

The Cross Product of the vectors is perpendicular to the plane and is proportional to one of the desired unit vectors.

To make its length equal to one, we simply divide by its magnitude:

.

Find 2 unit vectors perpendicular to **a**= 2 **i** - **j** + 3**k** and **b** = -4**i** + 2**j** - **k**.

Solution

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ -4 & 2 & -1 \end{vmatrix} = (-5\vec{i} - 10\vec{j})$$

$$\left\| -5\vec{i} - 10\vec{j} \right\| = \sqrt{25+100} = \sqrt{125} = 5\sqrt{5}$$

Divide the vector by its magnitude

$$\frac{1}{5\sqrt{5}}\left(-5\vec{i} - 10\vec{j}\right) = \frac{-5}{5\sqrt{5}}\vec{i} - \frac{10}{5\sqrt{5}}\vec{j} = \frac{-1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{j} = \frac{-\sqrt{5}}{5}\vec{i} - \frac{2\sqrt{5}}{5}\vec{j}$$

For a second unit vector simply multiply the answer by -1 $\frac{\sqrt{5}}{5}\vec{i} + \frac{2\sqrt{5}}{5}\vec{j}$

CROSS PRODUCT

Find a vector perpendicular to the plane that passes through the points

P(1, 4, 6), *Q*(-2, 5, -1), *R*(1, -1, 1)

Objective 3

FIND THE AREA OF A PARALLELOGRAM GIVEN TWO VECTORS

Example for property 4

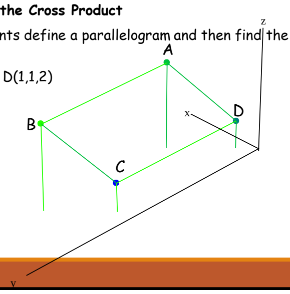
4. $\left\| \vec{u} \times \vec{v} \right\| = area\ of\ the\ paralleogram\ having$

$\vec{u}\ and\ \vec{v}\ as\ adjacent\ sides.$

Area of a Parallelogram via the Cross Product

Show that the following 4 points define a parallelogram and then find the area.

A(4,4,6), B(4,14,6), C(1,11,2), D(1,1,2)



Solution:
First we find the vectors of two adjacent sides:

$$\overrightarrow{AB} = \langle 4-4, 4-14, 6-6 \rangle = \langle 0, -10, 0 \rangle$$

$$\overrightarrow{AD} = \langle 4-1, 4-1, 6-2 \rangle = \langle 3, 3, 4 \rangle$$

Now find the vectors associated with the opposite sides:

$$\overrightarrow{DC} = \langle 1-1, 1-11, 2-2 \rangle = \langle 0, -10, 0 \rangle = \overrightarrow{AB}$$

$$\overrightarrow{DB} = \langle 1-4, 11-14, 2-6 \rangle = \langle -3, -3, -4 \rangle = -\overrightarrow{AD}$$

This shows that opposite sides are associated with the same vector, hence parallel. Thus the figure is of a parallelogram.

Continued

The area is equal to the magnitude of the cross product of vectors representing two adjacent sides:
Area = **|AB XAD|**

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -10 & 0 \\ 3 & 3 & 4 \end{vmatrix} = -40\vec{i} + 0\vec{j} + 30\vec{k}$$

$$\left\| -40\vec{i} + 30\vec{k} \right\| = \sqrt{(-40)^2 + 30^2} = \sqrt{1600 + 900} = \sqrt{2500} = 50$$

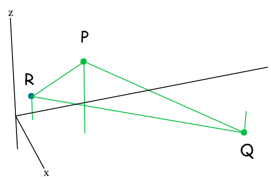
The area of the parallelogram is 50 square units.

Area of a Triangle via the Cross Product

Since the area of a triangle is based on the area of a parallelogram, it follows that the area would be ½ of the cross product of vectors of two adjacent sides.

Find the area of the triangle whose vertices are P(4,4,6), Q(5,16,-2) R(1,1,2)

Area parallelogram) = |**PQ** x **PR**|. The area of the triangle is half of this.



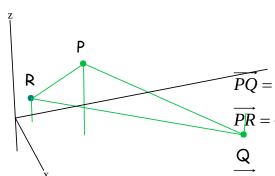
Area of a Triangle via the Cross Product Continued

P(4,4,6), Q(5,16,-2) R(1,1,2)

Area parallelogram) = |**PQ** x **PR**|

But what if we choose RP and RQ? Would the result be the same?

Let's do both and see!



$$\overrightarrow{PQ} = \langle 5-4, 16-4, -2-6 \rangle = \langle 1, 12, -8 \rangle$$

$$\overrightarrow{PR} = \langle 1-4, 1-4, 2-6 \rangle = \langle -3, -3, -4 \rangle$$

$$\overrightarrow{RP} = \langle 4-1, 4-1, 6-2 \rangle = \langle 3, 3, 4 \rangle$$

$$\overrightarrow{RQ} = \langle 5-1, 16-1, -2-2 \rangle = \langle 4, 15, -4 \rangle$$

$$\overrightarrow{PQ} = \langle 5-4, 16-4, -2-6 \rangle = \langle 1, 12, -8 \rangle \qquad \overrightarrow{RP} = \langle 4-1, 4-1, 6-2 \rangle = \langle 3, 3, 4 \rangle$$

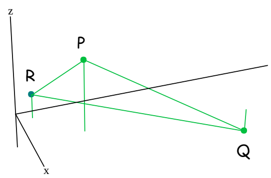
$$\overrightarrow{PR} = \langle 1-4, 1-4, 2-6 \rangle = \langle -3, -3, -4 \rangle \qquad \overrightarrow{RQ} = \langle 5-1, 16-1, -2-2 \rangle = \langle 4, 15, -4 \rangle$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 12 & -8 \\ -3 & -3 & -4 \end{vmatrix} = (-48-24)\vec{i} - (-4-24)\vec{j} + (-3+36)\vec{k} = -72\vec{i} + 28\vec{j} + 33\vec{k}$$

$$\frac{1}{2}\left\| -72\vec{i} + 28\vec{j} + 33\vec{k} \right\| = \frac{1}{2}\sqrt{(-72)^2 + 28^2 + 33^2} = \frac{1}{2}\sqrt{5184 + 784 + 1089} = \frac{1}{2}\sqrt{7075}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & 4 \\ 4 & 15 & -4 \end{vmatrix} = \left((-12-60)\vec{i} - (-12-16)\vec{j} + (45-12)\vec{k} \right) = -72\vec{i} + 28\vec{j} + 33\vec{k}$$

Since this is the same vector, the magnitude would be the same also.



The area of the triangle is $\frac{1}{2}\sqrt{7057} \approx 42$ square units.

CROSS PRODUCT

Find the area of the triangle with vertices

$P(1, 4, 6)$, $Q(-2, 5, -1)$, $R(1, -1, 1)$

Objective 4

HOW TO FIND THE VOLUME OF A PARALLELEPIPED GIVEN THREE VECTORS

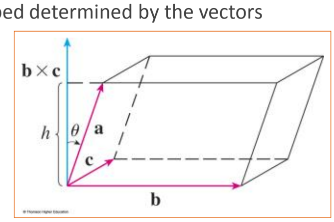
The triple scalar product is defined as: $\vec{u} \cdot \left(\vec{v} \times \vec{w}\right)$

For $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k},\ \ \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k},$ *and* $\ \vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$

$$\vec{u} \cdot \left(\vec{v} \times \vec{w}\right) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

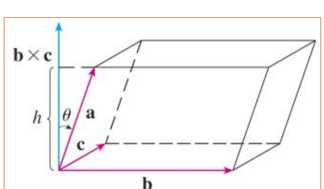
SCALAR TRIPLE PRODUCTS

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c**.



SCALAR TRIPLE PRODUCTS

The area of the base parallelogram is: *A* = |**b** x **c**|

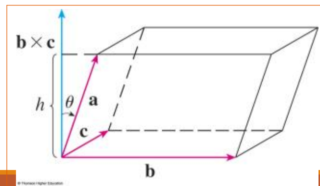


SCALAR TRIPLE PRODUCTS

If θ is the angle between **a** and **b** x **c**, then the height h of the parallelepiped is:

$$h = |\mathbf{a}||\cos\theta|$$

- We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.



SCALAR TRIPLE PRODUCTS

Hence, the volume of the parallelepiped is:

$$\begin{aligned} V &= Ah \\ &= |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| \\ &= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \end{aligned}$$

- Thus, we have proved the following formula.

Formula 11

SCALAR TRIPLE PRODUCTS

The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Volume of a Parallelepiped via the Scalar Triple Product:
Find the volume of the parallelepiped with adjacent edges AB, AC, and AD, where the points are A(4, -3, -2), B(2, 0, 5), C(-3, 2, 1), and D(1, 3, 2).

Volume of a Parallelepiped via the Scalar Triple Product:
Find the volume of the parallelepiped with adjacent edges AB, AC, and AD, where the points are A(4, -3, -2), B(2, 0, 5), C(-3, 2, 1), and D(1, 3, 2).

The volume is given by the scalar triple product: **AB** · (**AC X AD**).
First we need the three vectors:
AB = [2 - 4]**i** + [0 - (-3)]**j** + [5 - (-2)]**k** = -2**i** + 3**j** + 7**k**.
AC = [-3 - 4]**j** + [2 - (-3)]**j** + [1 - (-2)]**k** = -7**i** + 5**j** + 3**k**
AD = [1 - 4]**i** + [3 - (-3)]**j** + [2 - (-2)]**k** = -3**i** + 6**j** + 4**k**
First, find the cross product:

$$\mathbf{AC} \times \mathbf{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & 5 & 3 \\ -3 & 6 & 4 \end{vmatrix}$$

= **i**[20 - 18] - **j**[-28 - (-9)] + **k**[-42 - (-15)] = 2**i** + 19**j** - 27**k**
Now form the dot product to get the volume:
Volume = |**AB** · (2**i** + 19**j** - 27**k**)|
= | (-2**i** + 3**j** + 7**k**) · (2**i** + 19**j** - 27**k**)|
= | 4 + 57 - 189 | = 136 cubic units