IBHL Lesson 80

VECTORS AND THE GEOMETRY OF SPACE – Dot (Scalar) Product)

VECTORS

So far, we have added two vectors and multiplied a vector by a scalar.

VECTORS

The question arises:

- Is it possible to multiply two vectors so that their product is a useful quantity?

One such product is the dot product, which we will discuss in this section.

VECTORS

Another is the cross product, which we will discuss in Lesson 81

VECTORS

Lesson 80

The Dot Product

In this section, we will learn about:
Various concepts related to the dot product and its applications.

THE DOT PRODUCT — Definition 1

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product of **a** and **b** is the number $\mathbf{a} \cdot \mathbf{b}$ given by:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

DOT PRODUCT

Thus, to find the dot product of **a** and **b**, we multiply corresponding components and add.

SCALAR PRODUCT

The result is not a vector.

It is a real number, that is, a scalar.

- For this reason, the dot product is sometimes called the scalar product (or inner product).

DOT PRODUCT

Though Definition 1 is given for three-dimensional (3-D) vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle = a_1b_1 + a_2b_2$$

DOT PRODUCT Example 1

$\langle 2, 4\rangle \cdot \langle 3, -1\rangle =$

$\langle -1, 7, 4\rangle \cdot \langle 6, 2, -\tfrac{1}{2}\rangle =$

$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) =$

DOT PRODUCT Example 1

$\langle 2, 4\rangle \cdot \langle 3, -1\rangle = 2(3) + 4(-1) = 2$

$\langle -1, 7, 4\rangle \cdot \langle 6, 2, -\tfrac{1}{2}\rangle = (-1)(6) + 7(2) + 4(-\tfrac{1}{2})$
$= 6$

$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1)$
$= 7$

DOT PRODUCT

The dot product obeys many of the laws that hold for ordinary products of real numbers.

- These are stated in the following theorem.

PROPERTIES OF DOT PRODUCT Theorem 2

If **a**, **b**, and **c** are vectors in R^3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $0 \cdot \mathbf{a} = 0$

DOT PRODUCT PROPERTIES

These properties are easily proved using Definition 1.

- For instance, the proofs of Properties 1 and 3 are as follows.

DOT PRODUCT PROPERTY 1 **Proof**

$\mathbf{a} \cdot \mathbf{a}$

$= a_1^2 + a_2^2 + a_3^2$

$= |\mathbf{a}|^2$

DOT PRODUCT PROPERTY 3 **Proof**

$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$

$= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$

$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$

$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$

$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$

$= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

GEOMETRIC INTERPRETATION

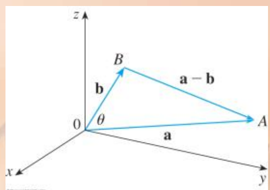
The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the angle θ between **a** and **b.**

- This is defined to be the angle between the representations of **a** and **b** that start at the origin, where $0 \le \theta \le \pi$.

GEOMETRIC INTERPRETATION

In other words, θ is the angle between the line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ here.

- Note that if **a** and **b** are parallel vectors, then $\theta = 0$ or $\theta = \pi$.



DOT PRODUCT

The formula in the following theorem is used by physicists as the definition of the dot product.

DOT PRODUCT—DEFINITION Theorem 3

If θ is the angle between the vectors **a** and **b**, then

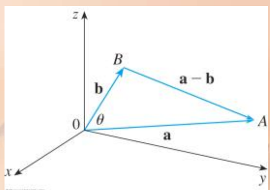
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

DOT PRODUCT—DEFINITION Proof—Equation 4

If we apply the Law of Cosines to triangle OAB here, we get:

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos \theta$$

- Observe that the Law of Cosines still applies in the limiting cases when $\theta = 0$ or π, or $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$



DOT PRODUCT—DEFINITION Proof

However,

$$|OA| = |\mathbf{a}|$$

$$|OB| = |\mathbf{b}|$$

$$|AB| = |\mathbf{a} - \mathbf{b}|$$

DOT PRODUCT—DEFINITION Proof—Equation 5

So, Equation 4 becomes:

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta$$

DOT PRODUCT—DEFINITION **Proof**

Using Properties 1, 2, and 3 of the dot product, we can rewrite the left side of the equation as follows:

$$\begin{aligned}|\mathbf{a}-\mathbf{b}|^2 &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b} \\ &= |\mathbf{a}|^2-2\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2\end{aligned}$$

DOT PRODUCT—DEFINITION **Proof**

Therefore, Equation 5 gives:

$$|\mathbf{a}|^2-2\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2 = |\mathbf{a}|^2+|\mathbf{b}|^2-2|\mathbf{a}||\mathbf{b}|\cos\theta$$

- Thus,

$$-2\mathbf{a}\cdot\mathbf{b} = -2|\mathbf{a}||\mathbf{b}|\cos\theta$$

or

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

DOT PRODUCT **Example 2**

If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a}\cdot\mathbf{b}$.

DOT PRODUCT **Example 2**

If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

- Using Theorem 3, we have:

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos(\pi/3) \\ &= 4 \cdot 6 \cdot \tfrac{1}{2} \\ &= 12\end{aligned}$$

DOT PRODUCT

The formula in Theorem 3 also enables us to find the angle between two vectors.

NONZERO VECTORS **Corollary 6**

If θ is the angle between the nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

NONZERO VECTORS Example 3

Find the angle between the vectors

a = ⟨2, 2, –1⟩ and **b** = ⟨5, –3, 2⟩

NONZERO VECTORS Example 3

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

and

$$|\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

Also,

$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$

NONZERO VECTORS Example 3

Thus, from Corollary 6, we have:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

- So, the angle between **a** and **b** is:

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84^\circ)$$

ORTHOGONAL VECTORS

Two nonzero vectors **a** and **b** are called perpendicular or orthogonal if the angle between them is $\theta = \pi/2$.

ORTHOGONAL VECTORS

Then, Theorem 3 gives:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/2) = 0$$

- Conversely, if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos \theta = 0$; so, $\theta = \pi/2$.

ZERO VECTORS

The zero vector **0** is considered to be perpendicular to all vectors.

- Therefore, we have the following method for determining whether two vectors are orthogonal.

ORTHOGONAL VECTORS **Theorem 7**

Two vectors **a** and **b** are orthogonal if and only if **a · b = 0**

ORTHOGONAL VECTORS **Example 4**

Show that 2**i** + 2**j** – **k** is perpendicular to 5**i** – 4**j** + 2**k**.

ORTHOGONAL VECTORS **Example 4**

Show that 2**i** + 2**j** – **k** is perpendicular to 5**i** – 4**j** + 2**k**.

- (2**i** + 2**j** – **k**) · (5**i** – 4**j** + 2**k**)
  = 2(5) + 2(–4) + (–1)(2)
  = 0
- So, these vectors are perpendicular by Theorem 7.

DOT PRODUCT

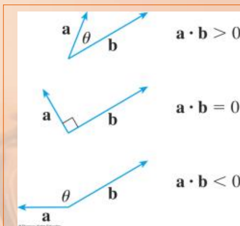
As $\cos\theta > 0$ if $0 \le \theta < \pi/2$ and $\cos\theta < 0$ if $\pi/2 < \theta \le \pi$, we see that **a** · **b** is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$.

- We can think of **a** · **b** as measuring the extent to which **a** and **b** point in the same direction.

DOT PRODUCT

The dot product **a** · **b** is:

- Positive, if **a** and **b** point in the same general direction
- Zero, if they are perpendicular
- Negative, if they point in generally opposite directions



DOT PRODUCT

In the extreme case where **a** and **b** point in exactly the same direction, we have $\theta = 0$.

- So, $\cos\theta = 1$ and **a** · **b** = |**a**||**b**|

DOT PRODUCT

If **a** and **b** point in exactly opposite directions, then $\theta = \pi$.

- So, $\cos \theta = -1$ and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}|\,|\mathbf{b}|$

APPLICATIONS OF PROJECTIONS

One use of projections occurs in physics in calculating work.

CALCULATING WORK

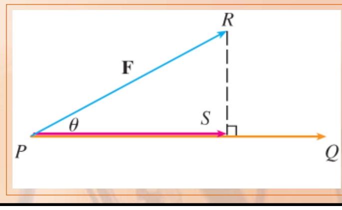
We defined the work done by a constant force F in moving an object through a distance d as:

$$W = Fd$$

- This, however, applies only when the force is directed along the line of motion of the object.

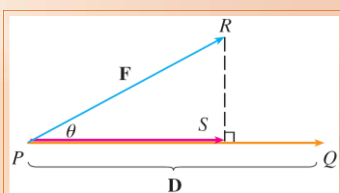
CALCULATING WORK

However, suppose that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as shown.



CALCULATING WORK

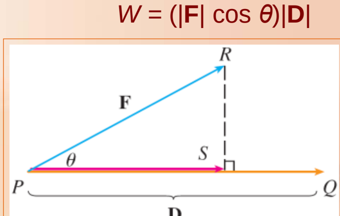
If the force moves the object from *P* to *Q*, then the displacement vector is $\mathbf{D} = \overrightarrow{PQ}$.



CALCULATING WORK

The work done by this force is defined to be the product of the component of the force along **D** and the distance moved:

$$W = (|\mathbf{F}| \cos \theta)|\mathbf{D}|$$



CALCULATING WORK **Equation 12**

However, from Theorem 3, we have:

$$W = |\mathbf{F}||\mathbf{D}| \cos \theta$$
$$= \mathbf{F} \cdot \mathbf{D}$$

CALCULATING WORK

Therefore, the work done by a constant force **F** is:

- The dot product **F** · **D**, where **D** is the displacement vector.

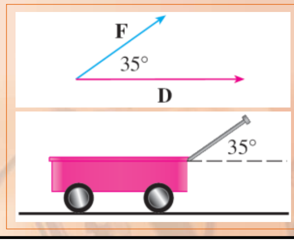
CALCULATING WORK **Example 7**

A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal.

- Find the work done by the force.

CALCULATING WORK Example 7

Suppose **F** and **D** are the force and displacement vectors, as shown.



CALCULATING WORK Example 7

Then, the work done is:

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos 35^\circ$$
$$= (70)(100) \cos 35^\circ$$
$$\approx 5734 \text{ N·m}$$
$$= 5734 \text{ J}$$

CALCULATING WORK Example 8

A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$.

- Find the work done.

CALCULATING WORK — Example 8

The displacement vector is $\mathbf{D} = \overrightarrow{PQ} = \langle 2, 5, 2 \rangle$

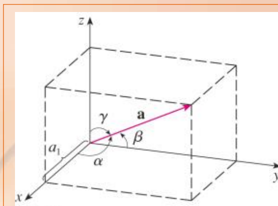
So, by Equation 12, the work done is:

$$W = \mathbf{F} \cdot \mathbf{D}$$
$$= \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle$$
$$= 6 + 20 + 10 = 36$$

- If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 joules.

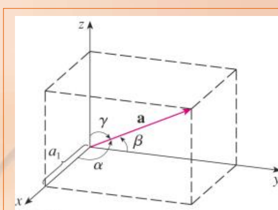
DIRECTION ANGLES

The direction angles of a nonzero vector **a** are the angles α, β, and γ (in the interval $[0, \pi]$) that **a** makes with the positive *x*-, *y*-, and *z*-axes.



DIRECTION COSINES

The cosines of these direction angles—cos α, cos β, and cos γ—are called the direction cosines of the vector **a**.



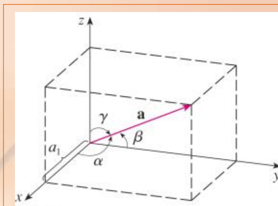
DIRECTION ANGLES & COSINES Equation 8

Using Corollary 6 with **b** replaced by **i**, we obtain:

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

DIRECTION ANGLES & COSINES

This can also be seen directly from the figure.



DIRECTION ANGLES & COSINES Equation 9

Similarly, we also have:

$$\cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

DIRECTION ANGLES & COSINES **Equation 10**

By squaring the expressions in Equations 8 and 9 and adding, we see that:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

DIRECTION ANGLES & COSINES

We can also use Equations 8 and 9 to write:

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3 \rangle \\ &= \langle |\mathbf{a}| \cos\alpha, |\mathbf{a}| \cos\beta, |\mathbf{a}| \cos\gamma \rangle \\ &= |\mathbf{a}| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle\end{aligned}$$

DIRECTION ANGLES & COSINES **Equation 11**

Therefore,

$$\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

- This states that the direction cosines of **a** are the components of the unit vector in the direction of **a**.

DIRECTION ANGLES & COSINES **Example 5**

Find the direction angles of the vector

a = ⟨1, 2, 3⟩

- $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$
- So, Equations 8 and 9 give:

$$\cos\alpha \frac{1}{\sqrt{14}} \qquad \cos\beta \frac{2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

DIRECTION ANGLES & COSINES **Example 5**

- Therefore,

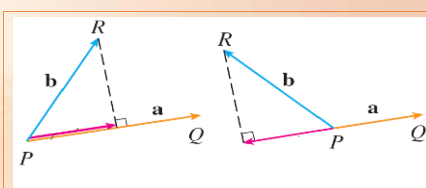
$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74^\circ$$

$$\beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ$$

$$\gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37^\circ$$

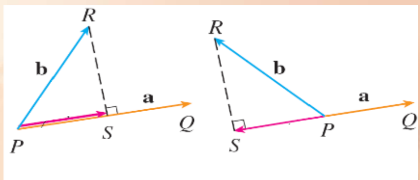
PROJECTIONS

The figure shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point *P*.



PROJECTIONS

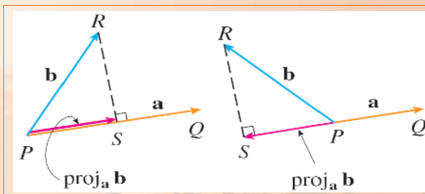
Let S be the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$.



VECTOR PROJECTION

Then, the vector with representation $\overrightarrow{PS}$ is called the vector projection of **b** onto **a** and is denoted by $\text{proj}_{\mathbf{a}}$ **b**.

- You can think of it as a shadow of **b**.



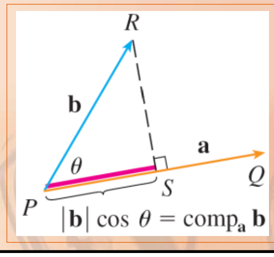
SCALAR PROJECTION

The scalar projection of **b** onto **a** (also called the component of **b** along **a**) is defined to be the signed magnitude of the vector projection.

PROJECTIONS

This is the number |**b**| cos θ, where θ is the angle between **a** and **b**.

- This is denoted by $\text{comp}_{\mathbf{a}}\,\mathbf{b}$.
- Observe that it is negative if $\pi/2 < \theta \le \pi$.



PROJECTIONS

The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = |\mathbf{a}|(|\mathbf{b}| \cos\theta)$$

shows that:

- The dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**.

PROJECTIONS

Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**.

PROJECTIONS

We summarize these ideas as follows.

PROJECTIONS

Scalar projection of **b** onto **a**:

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

Vector projection of **b** onto **a**:

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

- Notice that the vector projection is the scalar projection times the unit vector in the direction of **a**.

PROJECTIONS Example 6

Find the scalar and vector projections of:

b = ⟨1, 1, 2⟩ onto **a** = ⟨–2 , 3, 1⟩

PROJECTIONS Example 6

Since

$$|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$$

the scalar projection of **b** onto **a** is:

$$\text{comp}_{\text{a}}\, \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}}$$

$$= \frac{3}{\sqrt{14}}$$

PROJECTIONS Example 6

The vector projection is that scalar projection times the unit vector in the direction of **a**:

$$\text{proj}_{\text{a}}\, \mathbf{b} = \frac{3}{\sqrt{14}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a}$$

$$= \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$