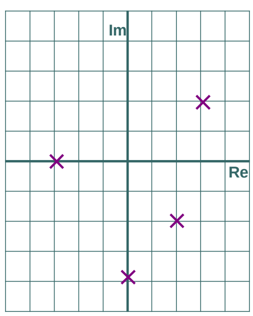
LESSON 72 – Geometric Representations of Complex Numbers

Argand Diagram
Modulus and Argument
Polar form

Argand Diagram

- Complex numbers can be shown Geometrically on an Argand diagram
- The real part of the number is represented on the x-axis and the imaginary part on the y.
  - -3
  - -4i
  - 3 + 2i
  - 2 – 2i

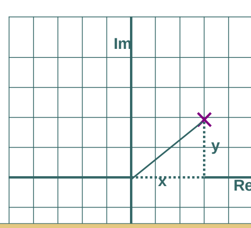


Modulus of a complex number

- A complex number can be represented by the position vector.

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

- The Modulus of a complex number is the distance from the origin to the point.



Can you generalise this?

$|z| = \sqrt{(x^2+y^2)}$

How many complex numbers in the form a + bi can you find with integer values of a and b that share the same modulus as the number above.

Could you mark all of the points?

What familiar shape would you draw?(more of LOCI later!)

Modulus questions

Find

a) $|3 + 4i|$

b) $|5 - 12i|$

c) $|6 - 8i|$

d) $|-24 - 10i|$

Find the distance between the first two complex numbers above. ***It may help to sketch a diagram***

Example

EXAMPLE 11.16

If $z = 1 + 2i$ and $w = x - i$, find (a) $|z + 4|$ (b) $|z + w|$

Find the minimum value of $|z + w|$.

Example - Solution

Solution

(a) First, we need to determine the complex number $z + 4$:

$z + 4 = (1 + 2i) + 4 = 5 + 2i$.

Then we have, $|5 + 2i| = \sqrt{25 + 4} = \sqrt{29}$

(b) First, we need to determine the complex number $z + w$:

$z + w = (1 + 2i) + (x - i) = (x + 1) + i \quad \therefore |(x + 1) + i| = \sqrt{(x + 1)^2 + 1}$.

$= \sqrt{x^2 + 2x + 2}$

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MATHEMATICS – Higher Level (Core)

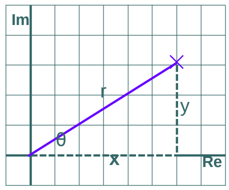
Now, $\sqrt{x^2 + 2x + 2} = \sqrt{(x^2 + 2x + 1) + 1} = \sqrt{(x + 1)^2 + 1}$. The minimum value will occur when $(x + 1)^2 + 1$ is a minimum. But, the minimum value of $(x + 1)^2 + 1$ is 1, therefore the minimum of $\sqrt{(x + 1)^2 + 1}$ is $\sqrt{1} = 1$.

The argument of a complex number

The argument of a complex number is the angle the line makes with the positive x-axis.

Can you generalise this?

$$\arg z = \theta$$
$$\theta = \tan^{-1}(y/x)$$
$$-\pi \le \theta \le \pi$$



It is really important that you sketch a diagram **before** working out the argument!!

The argument of a complex number

- Calculate the modulus and argument of the following complex numbers. (Hint, it helps to draw a diagram)
  1) 3 + 4i
  2) 5 – 5i
  3) -2√3 + 2i

The Polar form of a complex number

- So far we have plotted the position of a complex number on the Argand diagram by going horizontally on the real axis and vertically on the imaginary.
- This is just like plotting co-ordinates on an x,y axis
- However it is also possible to locate the position of a complex number by the distance travelled from the origin (pole), and the angle turned through from the positive x-axis.
- These are called "Polar coordinates"

The Polar form of a complex number

(x,y)

REAL Part

IMAGINARY part

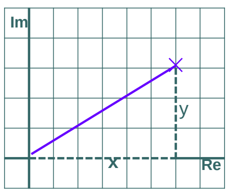
r is the MODULUS

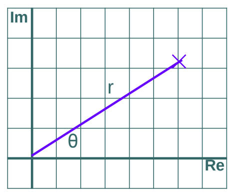
The ARGUMENT

(r, θ)

$\cos\theta = x/r$, $\sin\theta = y/r$

$x = r\cos\theta$, $y = r\sin\theta$,





Converting from Cartesian to Polar

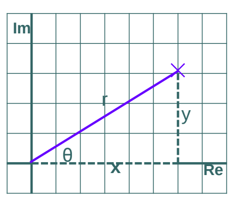
$$(x,y)=(r,\theta)=\left(\sqrt{x^2+y^2},\tan^{-1}\frac{y}{x}\right)$$

Convert the following from Cartesian to Polar

i) (1,1) =

ii) $(-\sqrt{3},1)$ =

iii) $(-4,-4\sqrt{3})$ =



Converting from Cartesian to Polar

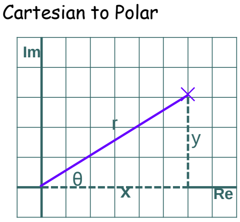
$$(x,y)=(r,\theta)=\left(\sqrt{x^2+y^2},\tan^{-1}\frac{y}{x}\right)$$

Convert the following from Cartesian to Polar

i) $(1,1) = (\sqrt{2},\pi/4)$

ii) $(-\sqrt{3},1) = (2,5\pi/6)$

iii) $(-4,-4\sqrt{3}) = (8,-2\pi/3)$



Converting from Polar to Cartesian

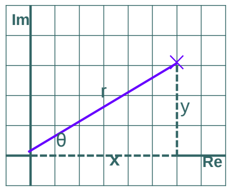
$$(r,\theta)=(x,y)(r\cos\theta, r\sin\theta)$$

Convert the following from Polar to Cartesian

i) (4,π/3) =

ii) (3√2,-π/4) =

iii) (6√2,3π/4) =



Converting from Polar to Cartesian

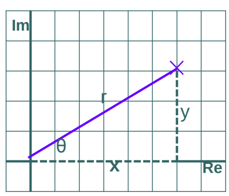
$$(r,\theta)=(x,y)(r\cos\theta, r\sin\theta)$$

Convert the following from Polar to Cartesian

i) (4,π/3) = (2,2√3)

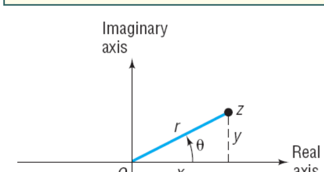
ii) (3√2,-π/4) = (3,-3)

iii) (6√2,3π/4) = (-6,6)



If $r \geq 0$ and $0 \leq \theta < 2\pi$, the complex number $z = x + yi$ may be written in **polar form** as

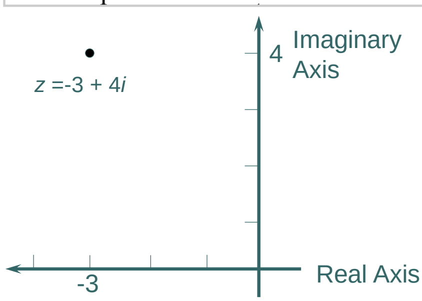
$$\boxed{z = x + yi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta) \qquad \textbf{(4)}}$$



$z = x + yi = r(\cos\theta + i\sin\theta)$,
$r \geq 0$, $0 \leq \theta < 2\pi$

- Plot the point corresponding to $z = -3 + 4i$ in the complex plane, and write an expression for z in polar form.

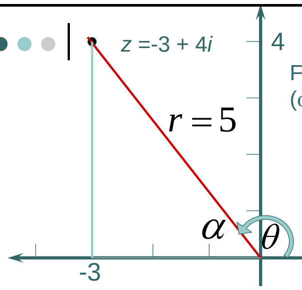
- Plot the point corresponding to $z = -3 + 4i$ in the complex plane, and write an expression for z in polar form.



- $z = -3 + 4i$ is in Quadrant II

  $x = -3$ and $y = 4$

$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$$



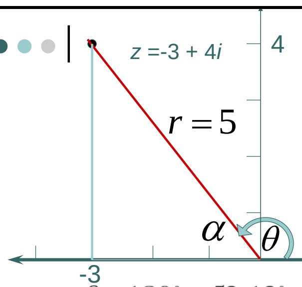
Find the reference angle (α) by solving

$$\tan \alpha = \left|\frac{y}{x}\right|$$

$$\tan \alpha = \left|\frac{4}{-3}\right|$$

$$\alpha = \tan^{-1}\frac{4}{3}$$

$$\alpha = 53.13°$$

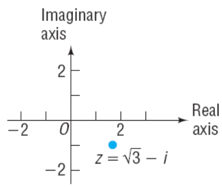


$$\theta = 180^\circ - 53.13^\circ = 126.87^\circ$$

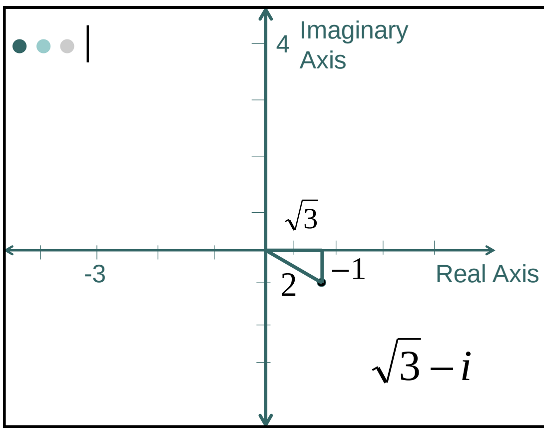
$$z = r(\cos\theta + i\sin\theta) =$$
$$= 5(\cos 126.87^\circ + i\sin 126.87^\circ)$$

Plot the point corresponding to $z = \sqrt{3} - i$ in the complex plane, and write an expression for z in polar form.

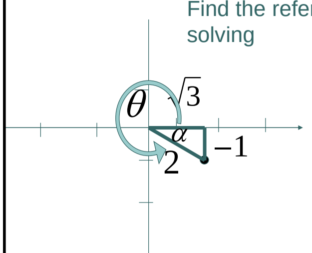
Plot the point corresponding to $z = \sqrt{3} - i$ in the complex plane, and write an expression for z in polar form.



Find r: $r = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2}$ $\quad r = 2$



Find the reference angle (α) by solving

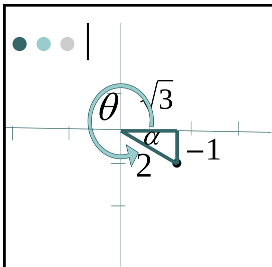


$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$\tan\alpha = \left|\frac{-1}{\sqrt{3}}\right|$$

$$\alpha = \tan^{-1}\left|\frac{-1}{\sqrt{3}}\right|$$

$$\alpha = 30^\circ$$



$$\theta = 360° - 30° = 330°$$
$$z = r(\cos\theta + i\sin\theta)$$
$$z = 2(\cos 330° + i\sin 330°)$$

Write in standard (rectangular) form.

$$2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

Write in standard (rectangular) form.

$$2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \qquad \sin\frac{5\pi}{6} = \frac{1}{2}$$

$$2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \qquad = -\sqrt{3} + i$$

Example:

Convert z = 3 cis 55° to rectangular form.

Convert z = -2 – 3*i* to polar form.

Example:

What is the absolute value of the following complex numbers:

$$z = 3 + 2i$$

$$z = 4\text{cis}\frac{2\pi}{3}$$

Multiply: $(3cis165^\circ)(4cis45^\circ)$

- Do you want to go thru that every time?

No!

$$(r\,cis\,\alpha)\cdot(t\,cis\,\beta)=r\,t\,cis\,(\alpha+\beta)$$

Multiply: $(4cis\,25^\circ)(6cis\,35^\circ)$

Divide: $\dfrac{3cis\,165^\circ}{4cis\,45^\circ}$

Properties of modulus

$|z_1 . z_2| = |z_1| . |z_2|$ $\quad |z^n| = |z|^n$

$|\bar{z}| = |z|$

$|z . \bar{z}| = |z|^2$ Proof : $z = a + ib, z\bar{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$|z_1 + z_2| \le |z_1| + |z_2|$ $\quad |z_1 + z_2| \ge ||z_1| - |z_2||$ (Triangle inequality)

$|z_1 - z_2| \ge ||z_1| - |z_2||$ $\quad |z_1 - z_2| \le |z_1| + |z_2|$

$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

Properties of Argument

$\mathrm{Arg}(z_1.z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$

$\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$

$\mathrm{Arg}(z_1.z_2....z_n) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) + ... + \mathrm{Arg}(z_n)$

$\mathrm{Arg}(\bar{z}) = -\,\mathrm{Arg}(z)$, $\mathrm{Arg}\left(\frac{1}{z}\right) = -\,\mathrm{Arg}(z)$

Arg(purely real) = 0 or π or $2n\pi$ and vice versa

Arg(purely imaginary) = $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ or $(2n+1)\frac{\pi}{2}$ and vice versa