Lesson 70

Further Work with Taylor and Maclaurin Series

In this section, we will learn:

(1) The series $(1 + x)^k$

(2) Error on Taylor Polynomials

TAYLOR SERIES

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

TAYLOR SERIES

For the special case *a* = 0, the Taylor series becomes:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$
$$=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots$$

MACLAURIN SERIES

This case arises frequently enough that it is given the special name Maclaurin series.

TAYLOR & MACLAURIN SERIES

Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

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Arranging our work in columns, we have:

$f(x) = (1+x)^k$	$f(0) = 1$
$f'(x) = k(1+x)^{k-1}$	$f'(0) = k$
$f''(x) = k(k-1)(1+x)^{k-2}$	$f''(0) = k(k-1)$
$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$	$f'''(0) = k(k-1)(k-2)$
.	.
.	.
.	.
$f^{(n)} = k(k-1)\cdots(k-n+1)(1+x)^{k-n}$	$f^{(n)}(0) = k(k-1)\cdots(k-n+1)$

BINOMIAL SERIES

Thus, the Maclaurin series of $f(x) = (1 + x)^k$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

- This series is called the binomial series.

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If its nth term is a_n, then

$$\left|\frac{a_{n+1}}{a_n}\right|$$

$$= \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|$$

$$= \frac{|k-n|}{n+1}|x| = \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x| \to |x| \qquad \text{as } n \to \infty$$

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Therefore, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$.

BINOMIAL COEFFICIENTS.

The traditional notation for the coefficients in the binomial series is:

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

- These numbers are called the binomial coefficients.

THE BINOMIAL SERIES

If k is any real number and $|x| < 1$, then

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty}\binom{k}{n}x^n \\ &= 1 + kx + \frac{k(k-1)}{2!}x^2 \\ &\quad + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots\end{aligned}$$

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Though the binomial series always converges when $|x| < 1$, the question of whether or not it converges at the endpoints, ± 1, depends on the value of k.

- It turns out that the series converges at 1 if $-1 < k \leq 0$ and at both endpoints if $k \geq 0$.

TAYLOR & MACLAURIN SERIES

Find the Maclaurin series for the function

$$f(x) = \frac{1}{\sqrt{4-x}}$$

and its radius of convergence.

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We write *f*(*x*) in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}}$$

$$= \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

TAYLOR & MACLAURIN SERIES

Using the binomial series with $k = -\frac{1}{2}$ and with x replaced by $-x/4$, we have:

$$\frac{1}{\sqrt{4-x}}$$

$$= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^{n}$$

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$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3 + \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x}{4}\right)^n + \cdots\right]$$

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$$= \frac{1}{2}\left[1 + \frac{1}{8}x + \frac{1\cdot 3}{2!8^2}x^2 + \frac{1\cdot 3\cdot 5}{3!8^3}x^3 + \cdots + \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}{n!8^n}x^n + \cdots\right]$$

- We know that this series converges when $|-x/4| < 1$, that is, $|x| < 4$.
- So, the radius of convergence is $R = 4$.

SUMMARY

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Further Examples

- Use the binomial series to expand the function as a power function. State the radius of convergence.

$$\text{(a) } y=\frac{1}{(1+x)^2}$$

$$\text{(b) } y=\frac{1}{\sqrt{2-x}}$$

Further Examples

- Use the binomial series to expand the function as a power function. State the radius of convergence.

$$\text{(a) } y=\sqrt{1+x}$$

$$\text{(b) } y=\frac{1}{(1+x)^4}$$

$$\text{(c) } y=\frac{1}{(2+x)^3}$$

$$\text{(d) } y=\sqrt[3]{(1-x)^2}$$

Lagrange Form (for error in Taylor Polynomials

Similar to the truncation error for an alternating series, finding the error using Taylor's Formula for the remainder is essentially given by the next term in the series:

$$R_n(x)=\frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}, a\le t\le x$$

EXAMPLE #1

- Approximate the value of ln(1.1) using a third degree Taylor polynomial and determine the maximum error in this approximation
- Choose f(x), choose "center" , take successive derivatives, evaluate
- To evaluate $R_3(1.1)$, choose a value of t that maximizes the error ($1 \leq t \leq 1.10$)

EXAMPLE #2

- Approximate cos(0.1) using a 4^{th} degree Taylor polynomial and find the associated LaGrange remainder, or error bound
- Choose f(x), choose "center" , take successive derivatives, evaluate
- To evaluate $R_4(0.1)$, choose a value of sin(t) or cos(t) that maximizes the error ($0 \leq t \leq x$)

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