Lesson 68 – Introduction to Taylor & Maclaurin Series

HL Math - Santowski

RECAP

- From last lesson:
- (1) We ***CAN*** model a function using a power series
- (2) These power series are based upon geometric series
- (3) Therefore, the functions that we can "work with" at this stage are variations of the function: $f(x) = \dfrac{1}{1-x}$
- Question → Can we develop a more "general" approach to developing a series in order to represent ANY function?

Suppose we wanted to find a fourth degree polynomial of the form: $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

that approximates the behavior of $f(x) = \ln(x+1)$ at $x = 0$

BUT HOW do we do that???

- We are going to start by making two assumptions:
- (a) Let's assume that the function y = f(x) does in fact have a power series representation about x = a (in this case, f(x) = ln(x + 1) at x = 0)
- (b) Next we will assume that the function y = f(x) has derivatives of every order and that we can in fact find them all

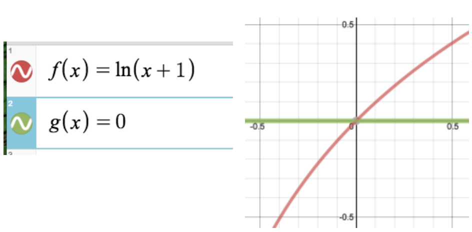
Suppose we wanted to find a fourth degree polynomial of the form: $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

that approximates the behavior of $f(x) = \ln(x+1)$ at $x = 0$

If we make $P(0) = f(0)$, and the first, second, third and fourth derivatives the same, then we would have a pretty good approximation.

$f(x) = \ln(x+1)$ $\quad P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$f(x) = \ln(x+1)$	$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$
$f(0) = \ln(1) = 0$	$P(0) = a_0 \longrightarrow \boxed{a_0 = 0}$



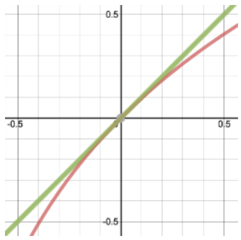
$f(x)=\ln(x+1)$ $\quad P(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$

$f'(x)=\frac{1}{1+x}$ $\quad P'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3$

$f'(0)=\frac{1}{1}=1$ $\quad P'(0)=a_1 \longrightarrow \boxed{a_1=1}$

$f(x)=\ln(x+1)$

$g(x)=0+1x$



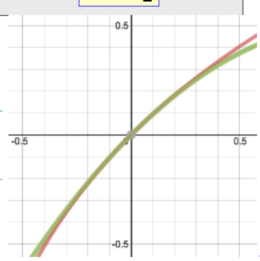
$f(x)=\ln(x+1)$ $\quad P(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$

$f''(x)=-\frac{1}{(1+x)^2}$ $\quad P''(x)=2a_2+6a_3x+12a_4x^2$

$f''(0)=-\frac{1}{1}=-1$ $\quad P''(0)=2a_2 \longrightarrow \boxed{a_2=-\frac{1}{2}}$

$f(x)=\ln(x+1)$

$g(x)=0+1x-\frac{1}{2}x^2$



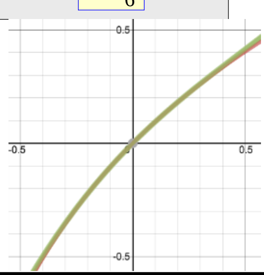
$f(x)=\ln(x+1)$ $\quad P(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$

$f'''(x)=2\cdot\frac{1}{(1+x)^3}$ $\quad P'''(x)=6a_3+24a_4x$

$f'''(0)=2$ $\quad P'''(0)=6a_3 \longrightarrow \boxed{a_3=\frac{2}{6}}$

$f(x)=\ln(x+1)$

$g(x)=0+1x-\frac{1}{2}x^2+\frac{2}{6}x^3$



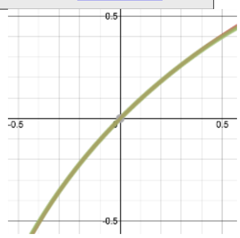
$f(x)=\ln(x+1)$ $P(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$

$f^{(4)}(x)=-6\frac{1}{(1+x)^4}$ $P^{(4)}(x)=24a_4$

$f^{(4)}(0)=-6$ $P^{(4)}(0)=24a_4 \rightarrow \boxed{a_4=-\frac{6}{24}}$

$f(x)=\ln(x+1)$

$g(x)=0+1x-\frac{1}{2}x^2+\frac{2}{6}x^3-\frac{6}{24}x^4$



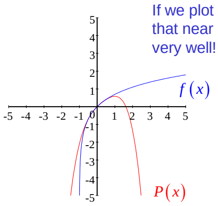
→

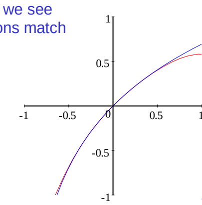
$P(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$ $f(x)=\ln(x+1)$

$P(x)=0+1x-\frac{1}{2}x^2+\frac{2}{6}x^3-\frac{6}{24}x^4$

$P(x)=0+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}$ $f(x)=\ln(x+1)$

If we plot both functions, we see that near zero the functions match very well!





→

Our polynomial: $0+1x-\frac{1}{2}x^2+\frac{2}{6}x^3-\frac{6}{24}x^4$

has the form: $f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{6}x^3+\frac{f^{(4)}(0)}{24}x^4$

or: $\frac{f(0)}{0!}+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$

This pattern occurs no matter what the original function was!

→

Maclaurin Series:

(generated by $f(x)$ at $x = 0$)

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

If we want to center the series (and it's graph) at some point other than zero, we get the Taylor Series:

Taylor Series:

(generated by $f(x)$ at $x = a$)

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

→

example: $y = \cos x$

$f(x) = \cos x \quad f(0) = 1$

$f'(x) = -\sin x \quad f'(0) = 0$

$f''(x) = -\cos x \quad f''(0) = -1$

$f'''(x) = \sin x \quad f'''(0) = 0$

$f^{(4)}(x) = \cos x \quad f^{(4)}(0) = 1$

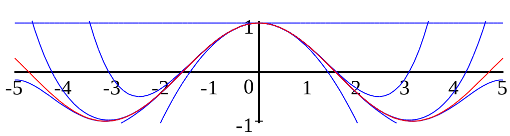
$$P(x) = 1 + 0x - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \frac{0x^5}{5!} - \frac{1x^6}{6!} + \cdots$$

$$\boxed{P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \cdots}$$

→

$y = \cos x$

$$P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \cdots$$



The more terms we add, the better our approximation.

→

example: $y=\cos(2x)$

Rather than start from scratch, we can use the function that we already know:

$$P(x)=1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\frac{(2x)^8}{8!}-\frac{(2x)^{10}}{10!}\cdots$$

→

example: $y=\cos(x)$ at $x=\frac{\pi}{2}$

$f(x)=\cos x \quad f\left(\frac{\pi}{2}\right)=0$

$f'(x)=-\sin x \quad f'\left(\frac{\pi}{2}\right)=-1$

$f''(x)=-\cos x \quad f''\left(\frac{\pi}{2}\right)=0$

$f'''(x)=\sin x \quad f'''\left(\frac{\pi}{2}\right)=1$

$f^{(4)}(x)=\cos x \quad f^{(4)}\left(\frac{\pi}{2}\right)=0$

$$P(x)=0-1\left(x-\frac{\pi}{2}\right)+\frac{0}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3+\cdots$$

$$\boxed{P(x)=-\left(x-\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}-\frac{\left(x-\frac{\pi}{2}\right)^5}{5!}+\cdots}$$

→

Further Examples

- Now let's work with:
- (1) y = sin(x)
- (2) y = e^x
- And develop the Taylor Polynomial for these two functions