Lesson 66 – Introduction to Power Series

Opening Examples

- Determine the sums of the following series:

(a) $S_\infty = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots$

(b) $S_\infty = 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \ldots\ldots$

(c) $S_\infty = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots\ldots$

(d) $S_\infty = 1 + x + x^2 + x^3 + \ldots\ldots$

Lesson Objectives

- The main goal of our lesson for today is to consider the sorts of functions that are sums of Power Series:
- What are these functions like?
- Are power series functions continuous? Are they differentiable? Antidifferentiable?
- Can we find formulas for them?

Opening exercise #1

- Expand the following series (6 – 8 terms) and comment upon the similarities and differences you notice:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{x^n}{n}$$
$$\text{(c)} \sum_{n=1}^{\infty} \frac{2^n}{n!} \qquad \text{(d)} \sum_{n=1}^{\infty} \frac{(2x)^n}{n!}$$
$$\text{(e)} \sum_{n=1}^{\infty} n \qquad \text{(f)} \sum_{n=1}^{\infty} nx^n$$
$$\text{(g)} \sum_{n=1}^{\infty} \frac{10^n}{(n+1)!} \qquad \text{(h)} \sum_{n=1}^{\infty} \frac{10^n x^n}{(n+1)!}$$

Opening exercise #2

- For the following series, write the first 5 terms and then test the convergences of the following series:
- :

$$\text{(a)} \sum_{k=1}^{\infty} \frac{2^k}{k!} \quad \text{(b)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \quad \text{(c)} \sum_{k=1}^{\infty} \frac{(-4)^k}{k!} \quad \text{(d)} \sum_{k=1}^{\infty} \frac{(0.5)^k}{k!} \quad \text{(e)} \sum_{k=1}^{\infty} \frac{214^k}{k!}$$

Opening exercise #2

- For the following series, write the first 5 terms and then test the convergences of the following series:

$$\text{(a)} \sum_{k=1}^{\infty} \frac{2^k}{k!} \quad \text{(b)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \quad \text{(c)} \sum_{k=1}^{\infty} \frac{(-4)^k}{k!} \quad \text{(d)} \sum_{k=1}^{\infty} \frac{(0.5)^k}{k!} \quad \text{(e)} \sum_{k=1}^{\infty} \frac{214^k}{k!}$$

- Make a general conclusion about the convergence of $\sum_{k=1}^{\infty} \frac{x^k}{k!}$ and prove that it is true:
- The series $\sum_{k=1}^{\infty} \frac{x^k}{k!}$ is an example of a POWER SERIES

Opening exercise #3

- Use DESMOS to:
- (1) graph $f(x)=\frac{1}{1-x}$
- (2) graph $\sum_{n=0}^{\infty} x^n$ so maybe use 100 as the upper limit rather than infinity
- (3) now expand $g(x)=\sum_{n=0}^{\infty} x^n = ...$ and call it $g(x)$
- (4) Evaluate & compare f(0.2) and g(0.2); f(-0.1) and g(-0.1); f(0.01) and g(0.01)

Formulas! Consider an (already familiar) Example

Our old friend the geometric series!

$$S_\infty = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+......$$

We know it converges to $S_\infty = \frac{a}{1-r} = \frac{1}{1-\left(\frac{1}{2}\right)} = 2$

Formulas! Consider an (already familiar) Example

And our NEW friend, also a geometric series!

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+\dots$$

And just exactly HOW the HECK is this a geometric series?

Formulas! Consider an (already familiar) Example

Our old friend the geometric series since $r = x$!

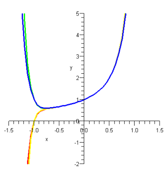
$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \ldots$$

We know it converges to $\frac{1}{1-x}$ whenever $|x| < 1$ and diverges elsewhere.

That is, $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for all x in $(-1,1)$.

Our first formula!

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for all } x \text{ in } (-1,1).$$



This plot shows the 10th, 12th, 13th, and 15th partial sums of this series.

We see the expected convergence on a "balanced" interval about $x = 0$.

Near $x = 1$, the partial sums "blow up" giving us the asymptote we expect to see there.

Near $x = -1$the even and odd partial sums go opposite directions, preventing any convergence to the left of $x = -1$.

Example #1

- Express $f(x) = \frac{1}{1+x}$ as a power series and find its interval of convergence

Example #1

EXAMPLE 1: Express $\frac{1}{1+x}$ as a power series and find the interval of convergence.

Solution: We have

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

Putting $u = -x$ in (1), we get

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n \implies \boxed{\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n}$$

with the interval of convergence

$$|-x| < 1 \implies |x| < 1 \implies \boxed{(-1,1)}$$

Example #2

- Express $f(x) = \frac{1}{5+x}$ as a power series and find its interval of convergence

Example #2

EXAMPLE 2: Express $\frac{1}{5+x}$ as a power series and find the interval of convergence.

Solution: We have

$$\frac{1}{5+x} = \frac{\frac{1}{5}}{1+\frac{x}{5}} = \frac{1}{5} \cdot \frac{1}{1+\frac{x}{5}} = \frac{1}{5} \cdot \frac{1}{1-\left(-\frac{x}{5}\right)}$$

Putting $u = -x/5$ in (1), we get

$$\frac{1}{1-\left(-\frac{x}{5}\right)} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^n} \implies \frac{1}{5} \cdot \frac{1}{1-\left(-\frac{x}{5}\right)} = \frac{1}{5} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^{n+1}}$$

Therefore

$$\boxed{\frac{1}{5+x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^{n+1}}}$$

with the interval of convergence

$$\left|-\frac{x}{5}\right| < 1 \implies \left|\frac{x}{5}\right| < 1 \implies |x| < 5 \implies \boxed{(-5,5)}$$

Example #3

- Express $f(x)=\dfrac{1}{1+x^2}$ as a power series and find its interval of convergence

Example #3

EXAMPLE 3: Express $\dfrac{1}{1+x^2}$ as a power series and find the interval of convergence.

Solution: We have

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}$$

Putting $u=-x^2$ in (1), we get

$$\frac{1}{1-(-x^2)}=\sum_{n=0}^{\infty}(-x^2)^n \quad\Longrightarrow\quad \boxed{\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}}$$

with the interval of convergence

$$|-x^2|<1 \quad\Longrightarrow\quad |x^2|<1 \quad\Longrightarrow\quad x^2<1 \quad\Longrightarrow\quad \boxed{(-1,1)}$$

Example #4

- Express $f(x)=\dfrac{1}{1+x^5}$ as a power series and find its interval of convergence

Example #4

EXAMPLE 4: Express $\frac{1}{1+x^5}$ as a power series and find the interval of convergence.

Solution: We have

$$\frac{1}{1+x^5} = \frac{1}{1-(-x^5)}$$

Putting $u = -x^5$ in (1), we get

$$\frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty}(-x^5)^n \implies \boxed{\frac{1}{1+x^5} = \sum_{n=0}^{\infty}(-1)^n x^{5n}}$$

with the interval of convergence

$$|-x^5| < 1 \implies |x^5| < 1 \implies |x| < 1 \implies \boxed{(-1,1)}$$

Example #5

- Express $f(x) = \frac{x^5}{7-9x^3}$ as a power series and find its interval of convergence

Example #5

EXAMPLE 5: Express $\frac{x^5}{7-9x^3}$ as a power series and find the interval of convergence.

Solution: We have

$$\frac{x^5}{7-9x^3} = \frac{\frac{x^5}{7}}{1-\frac{9x^3}{7}} = \frac{x^5}{7}\cdot\frac{1}{1-\frac{9x^3}{7}}$$

Putting $u = \frac{9x^3}{7}$ in (1), we get

$$\frac{1}{1-\frac{9x^3}{7}} = \sum_{n=0}^{\infty}\left(\frac{9x^3}{7}\right)^n = \sum_{n=0}^{\infty}\frac{9^n x^{3n}}{7^n} \implies \frac{x^5}{7}\cdot\frac{1}{1-\frac{9x^3}{7}} = \frac{x^5}{7}\cdot\sum_{n=0}^{\infty}\frac{9^n x^{3n}}{7^n} = \sum_{n=0}^{\infty}\frac{9^n x^{3n+5}}{7^{n+1}}$$

thus

$$\boxed{\frac{x^5}{7-9x^3} = \sum_{n=0}^{\infty}\frac{9^n x^{3n+5}}{7^{n+1}}}$$

with the interval of convergence

$$\left|\frac{9x^3}{7}\right| < 1 \implies |x^3| < \frac{7}{9} \implies \boxed{\left(-\sqrt[3]{\frac{7}{9}}, \sqrt[3]{\frac{7}{9}}\right)}$$

Example #6

- Express $f(x)=\dfrac{3}{4-x}$ as a power series and find its interval of convergence

Find a geometric power series for the function: $f(x)=\dfrac{3}{4-x}$

$$\sum_{n=0}^{\infty} ar^n=\frac{a}{1-r}$$

We write $\dfrac{3}{4-x}$ in the form $\dfrac{a}{1-r}$ and identify a and r

Make this 1

Divide numerator and denominator by 4

$$\frac{3}{4-x}=\frac{\frac{3}{4}}{\frac{4}{4}-\frac{x}{4}}=\frac{3/4}{1-x/4}$$

a = 3/4 **r = x/4**

Use a and r to write the power series.

$$f(x)=\sum_{n=0}^{\infty} ar^n=\sum_{n=0}^{\infty}\frac{3}{4}\left(\frac{x}{4}\right)^n=\sum_{n=0}^{\infty}\frac{3x^n}{4^{n+1}}=\frac{3}{4}+\frac{3}{16}x+\frac{3}{64}x^2+\ldots.\frac{3x^n}{4^{n+1}}\ldots..$$

Find a geometric power series for the function: $f(x)=\dfrac{3}{4-x}$

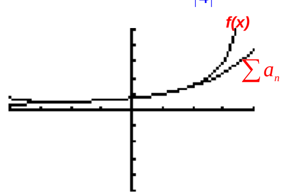
$$f(x)=\sum_{n=0}^{\infty}\frac{3x^n}{4^{n+1}}=\frac{3}{4}+\frac{3}{16}x+\frac{3}{64}x^2+\ldots.\frac{3x^n}{4^{n+1}}\ldots..$$

Let us obtain the interval of convergence for this power series.

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{3x^{n+1}}{4^{n+2}}\cdot\frac{4^{n+1}}{3x^n}\right|=\left|\frac{x}{4}\right|$$

The series converges for $\left|\dfrac{x}{4}\right|<1$

-4 < x < 4



Watch the graph of f(x) and the graph of the first four terms of the power series.

The convergence of the two on (-4, 4) is obvious.

Example #7

- Express $f(x)=\dfrac{3}{4-x}$ as a power series centered at x = -2 and find its interval of convergence

Find a geometric power series centered at c = -2 for the function: $f(x)=\dfrac{3}{4-x}$

The power series centered at c is: $\sum_{n=0}^{\infty} a(r-c)^n = \dfrac{a}{1-(r-c)}$

We write $\dfrac{3}{4-x}$ in the form $\dfrac{a}{1-(r-c)} = \dfrac{3}{4-[x-(-2)]+2} = \dfrac{3}{6-(x+2)}$

Divide numerator and denominator by 6

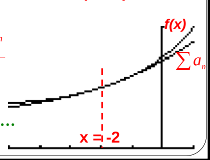
Make this 1

$$\frac{\frac{3}{6}}{\frac{6}{6}-\frac{(x+2)}{6}} = \frac{1/2}{1-(x+2)/6}$$

a = ½   r – c = (x + 2)/6

$$f(x)=\sum_{n=0}^{\infty} a(r-c)^n = \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{x+2}{6}\right)^n = \sum_{n=0}^{\infty}\frac{(x+2)^n}{2\cdot 6^n}$$

$$=\frac{1}{2}+\frac{x+2}{2\cdot 6}+\frac{(x+2)^2}{2\cdot 6^2}+\ldots\ldots\frac{(x+2)^n}{2\cdot 6^n}\ldots.$$



Example #8

- Express $f(x)=\dfrac{3}{2x-1}$ as a power series centered at x = 2 and find its interval of convergence

Find a geometric power series centered at c = 2 for the function: $f(x)=\frac{3}{2x-1}$

We write $\frac{3}{2x-1}$ in the standard form $\frac{a}{1-r}$

$$\frac{3}{2x-1}=\frac{3}{-1+2x}=\frac{3}{-1+2(x-2)+4}$$

Since c = 2, this x has to become x – 2

Add 4 to compensate for subtracting 4

$$=\frac{3}{3+2(x-2)}$$

Divide by 3 to make this 1

$$=\frac{3/3}{\frac{3}{3}+\frac{2}{3}(x-2)}=\frac{1}{1+\frac{2}{3}(x-2)}$$

Make this – to bring it to standard form

$$=\frac{1}{1-\left(-\frac{2}{3}(x-2)\right)}$$

Find a geometric power series centered at c = 2 for the function: $f(x)=\frac{3}{2x-1}$

$$f(x)=\frac{3}{2x-1}=\frac{1}{1-\left(-\frac{2}{3}(x-2)\right)}$$

$a=1 \qquad r-c=-\frac{2}{3}(x-2)$

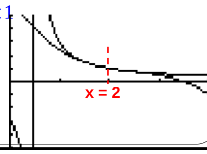
$$f(x)=\sum_{n=0}^{\infty}a(r-c)^n=\sum_{n=0}^{\infty}1\left[-\frac{2}{3}(x-2)\right]^n=\sum_{n=0}^{\infty}(-1)^n\frac{2^n(x-2)^n}{3^n}$$

$$=1-\frac{2(x-2)}{3}+\frac{2^2(x-2)^2}{3^2}-\frac{2^3(x-2)^3}{3^3}+\ldots\ldots+(-1)^n\frac{2^n(x-2)^n}{3^n}\ldots\ldots$$

This series converges for: $\left|-\frac{2}{3}(x-2)\right|<1$

$|(x-2)|<\frac{3}{2}$

$-\frac{3}{2}<(x-2)<\frac{3}{2} \qquad \frac{1}{2}<x<\frac{7}{2}$



Example #9

- Express $f(x)=\frac{4}{4+x^2}$ as a power series and find its interval of convergence

Find a geometric power series centered at c = 0 for the function: $f(x) = \dfrac{4}{4+x^2}$

We first obtain the power series for 4/(4 + x) and then replace x by x²

$$\frac{4}{4+x} = \frac{4/4}{\frac{4}{4}+\frac{x}{4}} = \frac{1}{1+\frac{1}{4}x} = \frac{1}{1-\left(-\frac{1}{4}x\right)} \qquad a=1 \qquad r=-\frac{1}{4}x$$

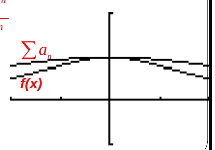
Make this – to bring it to standard form

Replace x by x²

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 1\left(-\frac{1}{4}x\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{4^n}$$

$$f(x) = \frac{4}{4+x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^n}{4^n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n}$$

$$= 1 - \frac{x^2}{4^2} + \frac{x^4}{4^4} - \frac{x^6}{4^6} + \,.....\, + (-1)^n \frac{x^{2n}}{4^n} .....$$



The series converges for $|x^2| < 4$

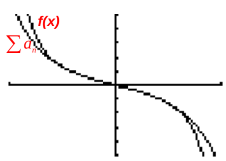
$-2 < x < 2$

Find a geometric power series centered at c = 0 for the function: $h(x) = \dfrac{2x}{x^2-1} = \dfrac{1}{1+x} - \dfrac{1}{1-x}$

We obtain power series for 1/(1 + x) and 1/(1 – x) and combine them.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} 1(-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$



$$h(x) = \frac{2x}{x^2-1} = \frac{1}{1+x} - \frac{1}{1-x}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^n\left[(-1)^n - 1\right]$$

$$= -2x - 2x^3 - 2x^5 - \,.....\, = \sum_{n=0}^{\infty} -2\,x^{2n+1} = \sum_{n=0}^{\infty} -2x^{2n+1}$$

STOP HERE!

Example #10 *****

- Express $f(x) = \dfrac{1}{(1-x)^2}$ as a power series and find its interval of convergence

Our first formula!

$$f(x)=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}\text{ for all } x \text{ in } (-1,1).$$

What else can we observe?

Clearly this function is both continuous and differentiable on its interval of convergence.

It is very tempting to say that the **derivative** for $f(x) = 1 + x + x^2 + x^3 + \ldots$

should be $1+2x+3x^2+4x^3+5x^4+\ldots$

But is it? For that matter, does this series even converge? And if it does converge, what does it converge to?

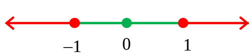
The general form of the series is

$$1+2x+3x^2+4x^3+\ldots=\sum_{n=0}^{\infty}(n+1)x^n$$

The ratio test limit:

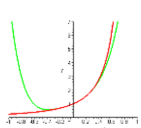
$$\lim_{n\to\infty}\frac{(n+2)|x|^{n+1}}{(n+1)|x|^n}=|x|\lim_{n\to\infty}\frac{(n+2)}{(n+1)}=|x|<1$$

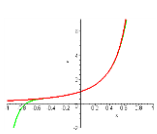
So the "derivative" series also converges on (-1,1). We showed that it diverges at the endpoints.



Differentiating Power Series

Does it converge to $\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{1}{(1-x)^2}$?


6th partial sum


10th partial sum

The green graph is the partial sum, the red graph is $\frac{1}{(1-x)^2}$

This graph suggests a general principle:

Theorem: (Derivatives and Antiderivatives of Power Series)

Let $S(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + a_4(x-x_0)^4 + \dots$
be a power series with radius of convergence $R > 0$.

And let $D(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + 4a_4(x-x_0)^3 + \dots$

And $A(x) = a_0(x-x_0) + \frac{a_1}{2}(x-x_0)^2 + \frac{a_2}{3}(x-x_0)^3 + \frac{a_3}{4}(x-x_0)^4 + \dots$

Then
- Both D and A converge with radius of convergence R.
- On the interval $(x_0 - R, x_0 + R)$ $S'(x) = D(x)$.
- On the interval $(x_0 - R, x_0 + R)$ $A'(x) = S(x)$.

Or to put it more succinctly, if a little less precisely,

Theorem: (Derivatives and Antiderivatives of Power Series)

If $S(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + a_4(x-x_0)^4 + \dots$

is a power series with radius of convergence $R > 0$.
Then we can differentiate and antidifferentiate S. Moreover,

$S'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + 4a_4(x-x_0)^3 + \dots$

And $\int S(x)\,dx = a_0(x-x_0) + \frac{a_1}{2}(x-x_0)^2 + \frac{a_2}{3}(x-x_0)^3 + \frac{a_3}{4}(x-x_0)^4 + \dots + C$

These all have the same radius of convergence.

Lest we lose the forest for the trees. . .

Let us consider again our original example from SLIDE #4

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k k^2}(x-3)^k.$$

Even though we can't find a formula for f, we can still differentiate and antidifferentiate it. What do we get?

$$f'(x) = \sum_{k=1}^{\infty} \frac{k}{2^k k^2}(x-3)^{k-1} = \sum_{k=1}^{\infty} \frac{1}{2^k k}(x-3)^{k-1}$$

$$\int f(x)\,dx = \sum_{k=1}^{\infty} \frac{1}{2^k k^2}\frac{(x-3)^{k+1}}{k+1} + C = \sum_{k=1}^{\infty} \frac{1}{2^k k^2 (k+1)}(x-3)^{k+1} + C$$

Lest we lose the forest for the trees. . .

Interval of Conv. $f(x)=\sum_{k=1}^{\infty}\frac{1}{2^k k^2}(x-3)^k$.

$$f'(x)=\sum_{k=1}^{\infty}\frac{k}{2^k k^2}(x-3)^{k-1}=\sum_{k=1}^{\infty}\frac{1}{2^k k}(x-3)^{k-1}$$

What is the radius of convergence?

$$\int f(x)\,dx=\sum_{k=1}^{\infty}\frac{1}{2^k k^2}\frac{(x-3)^{k+1}}{k+1}+C=\sum_{k=1}^{\infty}\frac{1}{2^k k^2(k+1)}(x-3)^{k+1}+C$$

What is the radius of convergence?

Example #11

- Express $f(x)=\ln(1+x)$ as a power series and find its interval of convergence

Example #12

- Express $f(x)=\ln(1-x^2)$ as a power series and find its interval of convergence

Find a geometric power series centered at c = 0 for the function: $f(x)=\ln(1-x^2)=\int\frac{1}{1+x}dx-\int\frac{1}{1-x}dx$

We obtain power series for 1/(1 + x) and integrate it

Then integrate the power series for 1/(1 – x) and combine both.

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}1(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n$$

$$\int\frac{1}{1+x}dx=\sum_{n=0}^{\infty}\int(-1)^n x^n dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}+c_1$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

$$\int\frac{1}{1-x}dx=\sum_{n=0}^{\infty}\int x^n dx=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}+c_2$$

Find a geometric power series centered at c = 0 for the function: $f(x)=\ln(1-x^2)=\int\frac{1}{1+x}dx-\int\frac{1}{1-x}dx$

$$\int\frac{1}{1+x}dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}+c_1 \qquad \int\frac{1}{1-x}dx=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}+c_2$$

$$\begin{aligned}f(x)=\ln(1-x^2)&=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}+c_1-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}+c_2\\&=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\left[(-1)^n-1\right]+c=-\frac{2x^2}{2}-\frac{2x^4}{4}-\frac{2x^6}{6}-.......+c\\&=\sum_{n=0}^{\infty}\frac{-2x^{2n+2}}{2n+2}+c=\sum_{n=0}^{\infty}\frac{(-1)x^{2n+2}}{n+1}+c\end{aligned}$$

$$\ln(1-x^2)=\sum_{n=0}^{\infty}\frac{(-1)x^{2n+2}}{n+1}+c$$

When x = 0, we have

0 = 0 + c **c = 0**

$$\ln(1-x^2)=-\sum_{n=0}^{\infty}\frac{x^{2n+2}}{n+1}$$

Example #13

- Express $f(x)=\ln(1+x^2)$ as a power series and find its interval of convergence

Find a geometric power series centered at c = 0 for f(x) = ln(x² + 1)

$$\frac{d}{dx}\ln(x^2+1)=\frac{2x}{x^2+1}$$

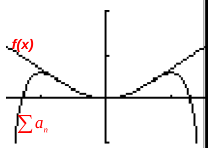
We obtain the power series for this and integrate.

$$\frac{2x}{x^2+1}=x\cdot\frac{2}{1-(-x^2)}=x\sum_{n=0}^{\infty}2(-1)^n x^{2n}=\sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}$$

$$\ln(x^2+1)=\sum_{n=0}^{\infty}\int(-1)^n 2x^{2n+1}dx=\sum_{n=0}^{\infty}(-1)^n\frac{2x^{2n+2}}{2n+2}+c$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{2x^{2n+2}}{2(n+1)}+c$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{n+1}+c$$



When x = 0, we have **0 = 0 + c** **c = 0**

$$\ln(x^2+1)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{n+1}$$

-1 < x < 1

Example #14

- Express $f(x)=\arctan(2x)$ as a power series and find its interval of convergence

Find a power series for the function f(x) = arctan 2x centered at c = 0

$$\frac{d}{dx}\arctan 2x=\frac{2}{1+4x^2}$$

We obtain the power series for this and integrate.

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-1)^n x^n$$

Replace x by 4x²

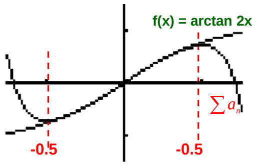
$$\frac{2}{1+4x^2}=2\sum_{n=0}^{\infty}(-1)^n\left(4x^2\right)^n$$

$$\arctan 2x=\int\frac{2}{1+4x^2}dx=2\sum_{n=0}^{\infty}\int(-1)^n\left(4x^2\right)^n dx=2\sum_{n=0}^{\infty}\int(-1)^n 4^n x^{2n}dx$$

$$=2\sum_{n=0}^{\infty}(-1)^n 4^n\frac{x^{2n+1}}{2n+1}+c$$

Find a power series for the function f(x) = arctan 2x centered at c = 0

$$\arctan 2x = 2\sum_{n=0}^{\infty}(-1)^n 4^n \frac{x^{2n+1}}{2n+1}+c$$
$$= 2\sum_{n=0}^{\infty}(-1)^n 2^{2n} \frac{x^{2n+1}}{2n+1}+c$$
$$= \sum_{n=0}^{\infty}(-1)^n 2^{2n+1} \frac{x^{2n+1}}{2n+1}+c$$



When x = 0, arctan 2x = 0 0 = 0 + c c = 0

$$\arctan 2x = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1} \frac{x^{2n+1}}{2n+1} = 2x - \frac{8x^3}{3} + \frac{32x^5}{5} - \frac{128x^7}{7} + ...$$

$$-\frac{1}{2} < x < \frac{1}{2}$$

Example

$$f(x) = \sum_{k=1}^{\infty}\frac{1}{2^k k^2}(x-3)^k$$

- Consider the power series function

What is the interval of convergence?

The ratio test shows that the radius of convergence is

?

So the series converges on

(?,?)

What about the endpoints?

I = ???

Example

- To see this idea "in action," Consider the power series

$$f(x) = \sum_{k=1}^{\infty}\frac{1}{2^k k^2}(x-3)^k.$$

What is the interval of convergence?

The ratio test shows that the radius of convergence is

2

So the series converges on

(1,5)

What about the endpoints?

I = [1,5]

The domain of the function $f(x) = \sum_{k=1}^{\infty}\frac{1}{2^k k^2}(x-3)^k$ is

Example—Graphs!

- Now use DESMOS to graph the 30th, 35th, 40th, and 45th partial sums of the power series

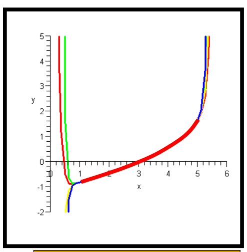
$$f(x)=\sum_{k=1}^{\infty}\frac{1}{2^k k^2}(x-3)^k.$$

Remember that each partial sum is a polynomial. So we can plot it!

Example—Graphs!

- This graph shows the 30th, 35th, 40th, and 45th partial sums of the power series

$$f(x)=\sum_{k=1}^{\infty}\frac{1}{2^k k^2}(x-3)^k.$$



Do you see the graph of *f* emerging in the picture?