Lesson 61 – Infinite Sequences

HL Math - Santowski

Lesson Objectives

- (1) Review basic concepts dealing with sequences
- (2) Evaluate the limits of infinite sequences
- (3) Understand basic concepts associated with limits of sequences
- (4) Introduce limits of functions & make the connection to infinite sequences

Setting the Stage

- Evaluate $\int e^{-x^2}\,dx$

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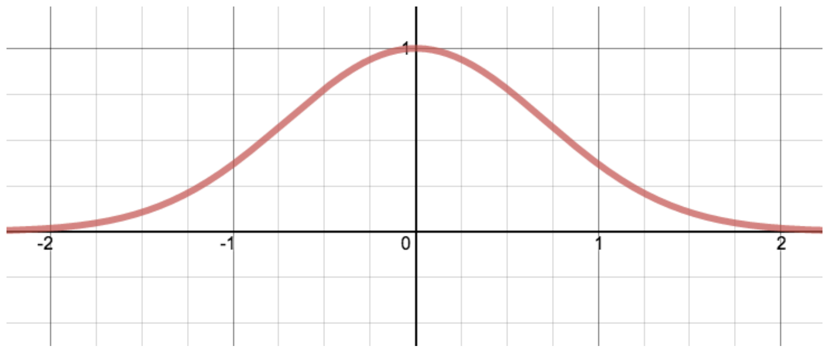
- Evaluate $\int e^{-x^2}\,dx$

- Explain why we can't evaluate this integral with the techniques discussed so far in this course

Explain the sequence in the following slides:

$$f(x) = e^{-x^2}$$



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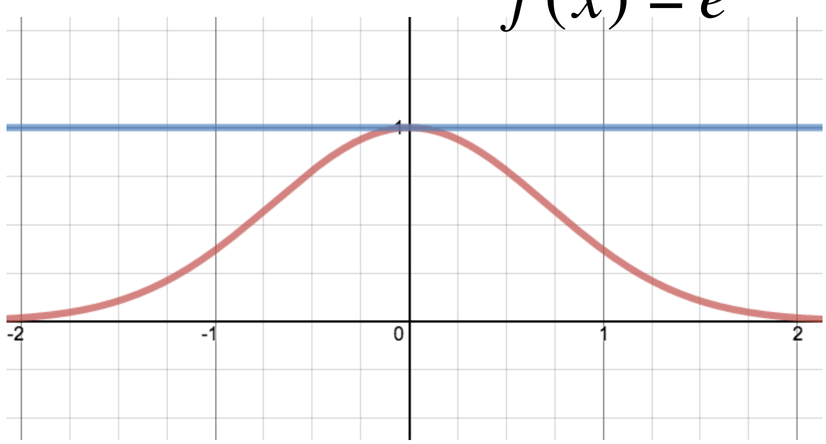

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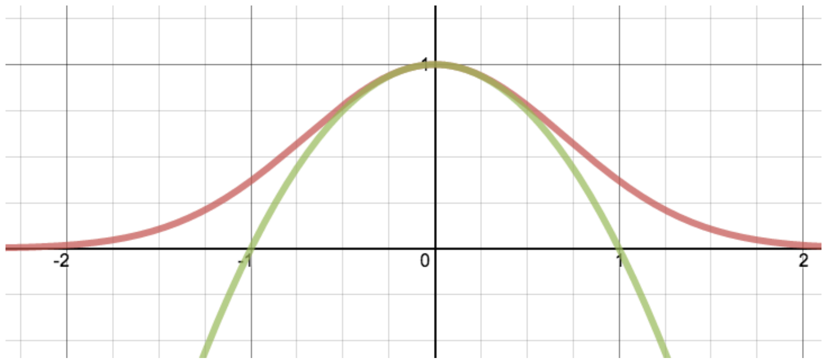

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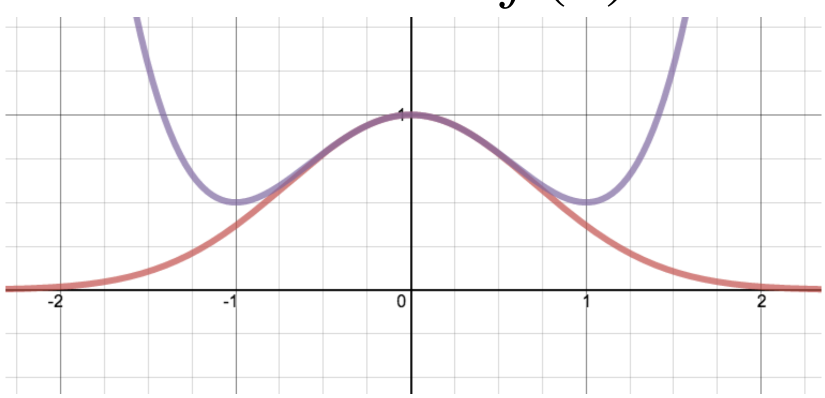

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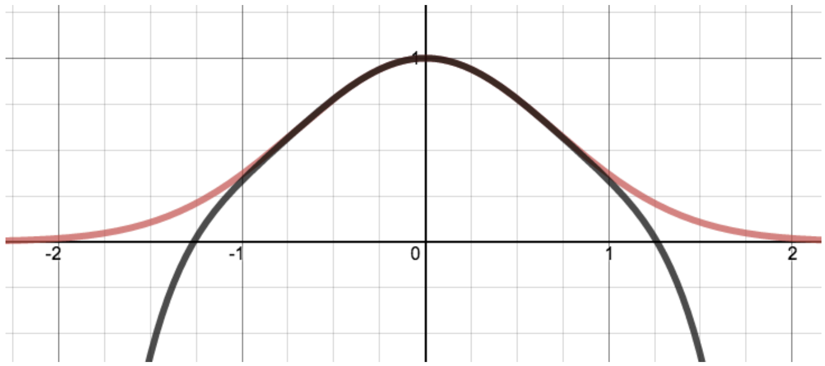

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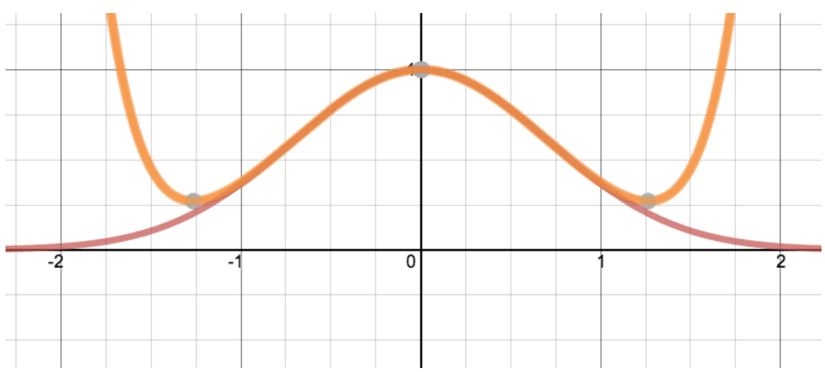

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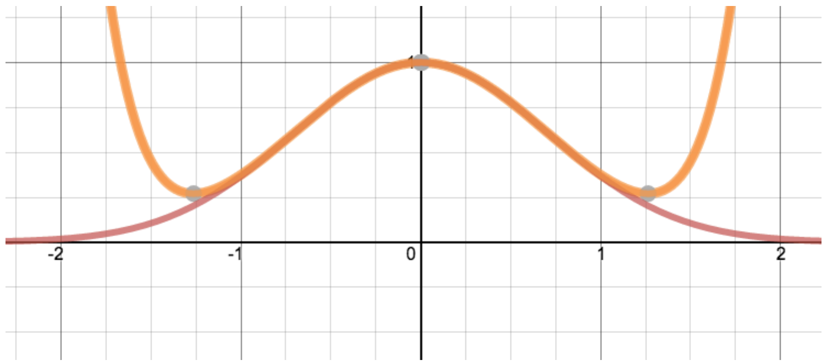


$$f(x) = 1 - \frac{1}{1!}x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - \ldots\ldots$$

(A) Review of Sequences

- List the first four terms of each of the following sequences:

$$\text{(a)} \left\{\frac{2n}{n+1}\right\}_{n=1}^{\infty}$$

$$\text{(b)} \left\{\sqrt{n+4}\right\}_{n=4}^{\infty}$$

$$\text{(c)} \left\{\sin\frac{n\pi}{6}\right\}_{n=1}^{\infty}$$

$$\text{(d)} \left\{\frac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$$

(A) Review of Sequences

- List the first four terms of each of the following sequences:

$$\text{(a) } u_{n+1} = 2\left(\sqrt{u_n} + 1\right) \text{ where } u_1 = 9$$

$$\text{(b) } u_{n+1} = 5 - 2u_n \text{ where } u_1 = -4$$

- Write an explicit expression for the general term of: $\left\{\frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, \frac{-6}{625}, \frac{7}{3125}, \ldots\ldots\right\}$

(B) Limits of a Sequence

- Investigate the behaviour of these sequences:

$$\text{(a) } u_n = \frac{4n^2}{2n^2 + 5}$$

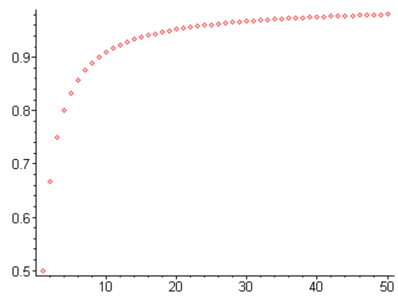
$$\text{(b) } u_n = \frac{4n^2}{2n + 5}$$

$$\text{(c) } u_n = \frac{(2n-1)!}{(2n+1)!}$$

$$\text{(d) } u_n = \frac{e^n + e^{-n}}{e^{2n} - 1}$$

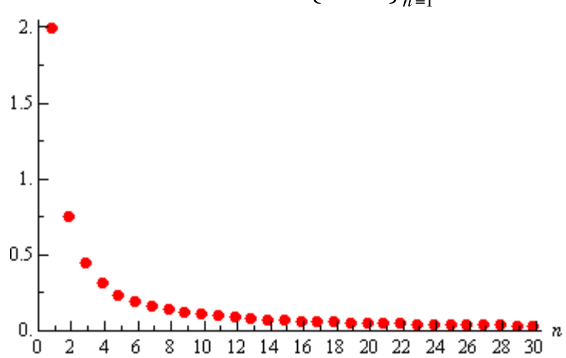
(B) Limit of a sequence

- Consider the sequence $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$
- If we plot some values we get this graph



(B) Limit of a sequence

- Consider the sequence $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$



(B) Limits of a Sequence

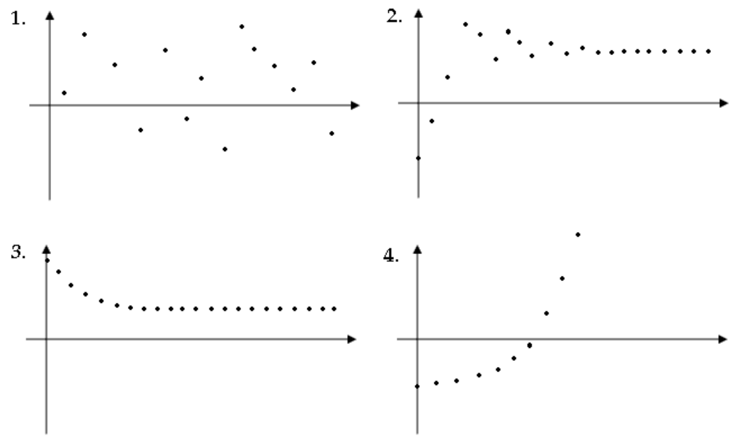
- We say that $\lim_{n \to \infty} a_n = L$ if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of the a_n's approach L as n approaches infinity.
- We say that $\lim_{n \to \infty} a_n = \infty$ if we can make a_n as large as we want for all sufficiently large n. Again, in other words, the value of the a_n's get larger and larger without bound as n approaches infinity.
- We say that $\lim_{n \to \infty} a_n = -\infty$ if we can make a_n as large and negative as we want for all sufficiently large n. Again, in other words, the value of the a_n's are negative and get larger and larger without bound as n approaches infinity.

(B) Limit of a sequence (Defn 1)

- A sequence $\{a_n\}$ has the limit ***L*** if we can make the terms of $\boldsymbol{a}_n$ as close to ***L*** as we like by taking *n* sufficiently large.
- We write

$$\lim_{n \to \infty} a_n = L \quad or \quad a_n \to L \; as \; n \to \infty$$

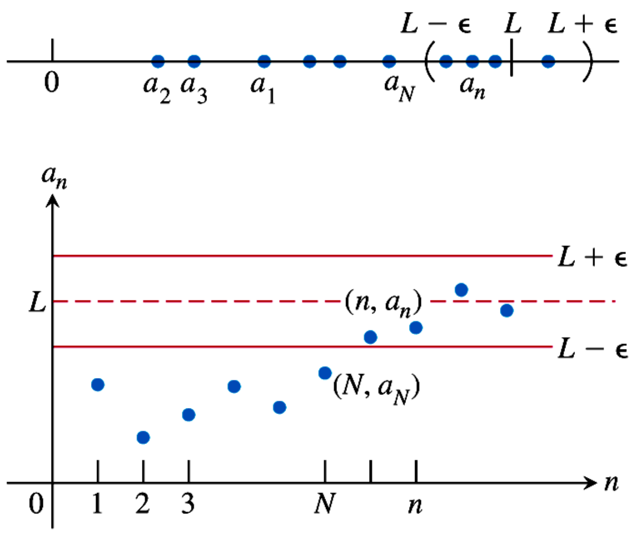
Limit of a sequence



(B) Limit of a sequence (Defn 2)

- A sequence $\{a_n\}$ has the limit ***L*** if for every $\varepsilon > 0$ there is a corresponding integer *N* such that $|a_n - L| < \varepsilon, \quad whenever\ n > N$
- We write $\lim_{n\to\infty} a_n = L \quad or \quad a_n \to L\ as\ n \to \infty$

y= L is a horizontal asymptote when sequence converges to L.



(B) Limits of a Sequence

- Example: Given the sequence $u_n = \dfrac{n+1}{2n+1}$
- (a) Find the minimum value of m such that
  $$n \geq m \Rightarrow \left| u_n - \frac{1}{2} \right| < 0.1$$
- (b) Consider the epsilon value of 0.001 and 0.00001. In each case, find the minimum value of m such that $n \geq m \Rightarrow \left| u_n - \dfrac{1}{2} \right| < \varepsilon$

Convergence/Divergence

- If $\lim_{n\to\infty} a_n$ exists we say that the sequence **converges**.
  - Note that for the sequence to converge, the limit must be finite
- If the sequence does not converge we will say that it **diverges**
  - Note that a sequence diverges if it approaches to infinity or if the sequence does not approach to anything

(B) Limits of a Sequence

- Which of the following sequences diverge or converge?

$$\text{(a) } u_n = \frac{n^3 + 2n}{n^2 + 4}$$

$$\text{(b) } u_n = \frac{3n}{n + 4}$$

$$\text{(c) } u_n = \frac{(-1)^n}{2^n}$$

$$\text{(d) } u_n = \sin(n)$$

Examples

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$$\lim_{n\to\infty}\frac{n+4}{n+1} \quad \lim_{n\to\infty}\frac{n-1}{n} \quad \lim_{n\to\infty}\frac{\ln n}{n} \quad \lim_{n\to\infty}\frac{n}{n^2}$$

$$\lim_{n\to\infty}\frac{220n^2}{n^2-4} \quad \lim_{n\to\infty}\frac{n!}{n^n} \quad \lim_{n\to\infty}e^{\frac{3n}{n+1}} \quad \lim_{n\to\infty}\sqrt{n+1}-\sqrt{n}$$

$$\lim_{n\to\infty}5^{\frac{1}{n-2}} \quad \lim_{n\to\infty}7^{-n} \quad \lim_{n\to\infty}\left(\frac{1}{8}\right)^n \quad \lim_{n\to\infty}\frac{\sqrt{n^2+1}}{2n}$$

$$\lim_{n\to\infty}n^{\frac{1}{n^2}} \quad \lim_{n\to\infty}\frac{\ln n}{n^\varepsilon} \quad \lim_{n\to\infty}\frac{n}{e^{\varepsilon n}} \quad \lim_{n\to\infty}\frac{10^n}{n!}$$

(C) More Limit Concepts

- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}\left(a_n \pm b_n\right) = \lim_{n\to\infty}a_n \pm \lim_{n\to\infty}b_n$$

$$\lim_{n\to\infty}\left(c\cdot a_n\right) = c\cdot \lim_{n\to\infty}a_n, \quad \lim_{n\to\infty}c = c$$

(C) More Limit Concepts

$$\lim_{n\to\infty}\left(a_n \cdot b_n\right) = \left(\lim_{n\to\infty} a_n\right)\cdot\left(\lim_{n\to\infty} b_n\right)$$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}, if\ \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty}\left(a_n{}^p\right) = \left(\lim_{n\to\infty} a_n\right)^p, if\ \ p > 0\ and\ a_n > 0$$

L'Hopital and sequences

- L'Hopital: Suppose that f(x) and g(x) are differentiable and that $g'(x) \neq 0$ near *a*. Also suppose that we have an indeterminate form of type $\frac{0}{0}\, or\, \frac{\infty}{\infty}$. Then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$

Squeeze Theorem for Sequences

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THEOREM 9.3 Squeeze Theorem for Sequences

If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then

$$\lim_{n\to\infty} c_n = L.$$

(C) More Limit Concepts

- Use the Squeeze theorem to investigate the convergence or divergence of:

$$\text{(a) } u_n = \frac{\sin(2n+1)}{n}$$

$$\text{(b) } u_n = \frac{\sin(n)}{n^2}$$

$$\text{(c) } \left\{\frac{\ln(n)}{n^2}\right\}_{n=1}^{\infty}$$

Definition of a Monotonic Sequence

Definition of a Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdot \cdot \cdot \leq a_n \leq \cdot \cdot \cdot$$

or if its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdot \cdot \cdot \geq a_n \geq \cdot \cdot \cdot .$$

Sequence

Definition of a Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** if there is a real number N such that $N \leq a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.

Monotonic …

- Consider the sequence $u_n = \left\{ \frac{2^n}{n!} \right\}_{n=1}^{\infty}$
- Determine whether the sequence is:
  - (a) increasing or decreasing
  - (b) bounded and if so, the max/min bound
  - (c) convergent

Monotonic …

- Graph BOTH $u_n = \frac{\ln(n)}{n^2}$ and $f(x) = \frac{\ln(x)}{x^2}$
- Consider the sequence $u_n = \frac{\ln(n)}{n^2}$. Show that this sequence is decreasing
- Is this sequence bounded?
- Does this sequence converge or diverge?

Bounded Monotonic Sequences

THEOREM 9.5 Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Video links patrick jmt

- https://www.youtube.com/watch?v=Kxh7yJC9Jr0
- https://www.youtube.com/watch?v=9K1xx6wfN-U
- Squeeze theorem
- https://youtu.be/TpbxFJphGyg
- https://youtu.be/jO5vIRvRrUU