Lesson 59 Rolle's Theorem and the Mean Value Theorem

HL Math - Calculus

After this lesson, you should be able to:

- **Understand and use Rolle's Theorem**
- **Understand and use the Mean Value Theorem**

Rolle's Theorem

If $f(a) = f(b)$ and if you connect from $f(a)$ to $f(b)$ with a smooth curve, there will be at least one place where $f'(c) = 0$

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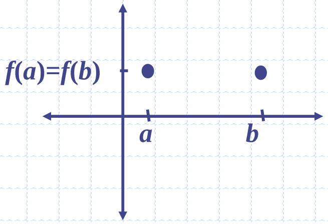
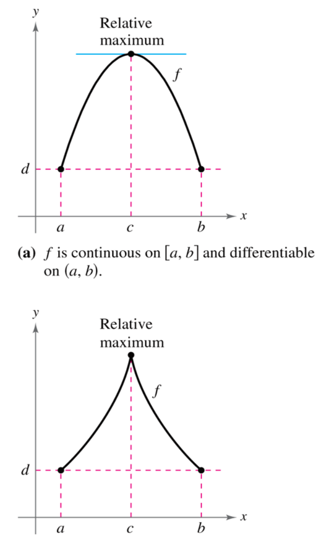


Rolle's Theorem

It just says that between any two points where the graph of the differentiable function $f(x)$ cuts the horizontal line there must be a point where $f'(x) = 0$. The following picture illustrates the theorem.



(a) f is continuous on $[a, b]$ and differentiable on (a, b).

(b) f is continuous on $[a, b]$.

Rolle's Theorem

If two points at the same height are connected by a continuous, differentiable function, then there has to be at least one place between those two points where the derivative, or slope, is zero.

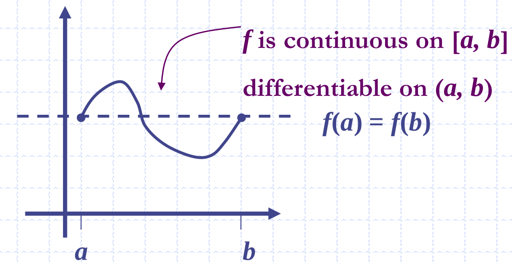
Rolle's Theorem

If 1) $f(x)$ is continuous on $[a, b]$,

2) $f(x)$ is differentiable on (a, b), and

3) $f(a) = f(b)$

then there is at least one value of x on (a, b), call it c, such that $f'(c) = 0$.



Example

Example 1 :

Does f(x) have any critical numbers between x = -2 and x = 2?

$$f(x) = x^4 - 2x^2 \quad \text{on}\,[-2, 2]$$

Example

<u>Example 1</u> $f(x) = x^4 - 2x^2$ on $[-2, 2]$

(f is continuous and differentiable)

$f(-2) = 8 = f(2)$

Since , then Rolle's Theorem applies…

$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 0$

then, $x = -1$, $x = 0$, and $x = 1$

Rolle's Theorem

Does Rolle's Theorem apply?

If not, why not?

If so, find the value of c.

<u>Example 2</u> $f(x) = 4 - x^2$ $[-2, 2]$

Rolle's Theorem

Does Rolle's Theorem apply?

If not, why not?

If so, find the value of *c*.

Example 3 $f(x) = x^3 - x$ $[-1, 1]$

Example

Example 4 $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ on $[-1, 1]$

(Graph the function over the interval on your calculator)

continuous on [-1, 1]

not differentiable at 0

not differentiable on (-1, 1)

***f*(-1) = 1 = *f*(1)**

Rolle's Theorem Does NOT apply since

Rolle's Theorem

Does Rolle's Theorem apply?

If not, why not?

If so, find the value of *c*.

Example 5 $f(x) = \dfrac{x^2 + 4}{x^2}$ $[-2, 2]$

Note

When working with Rolle's make sure you

1. State $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

2. Show that $f(a) = f(b)$.

3. State that there exists at least one $x = c$ in (a, b) such that $f'(c) = 0$.

This theorem only guarantees the existence of an extrema in an open interval. It does not tell you how to find them or how many to expect. If YOU can not find such extrema, it does not mean that it can not be found. In most of cases, it is enough to know the existence of such extrema.

Rolle's Theorem for Derivatives

Example: Determine whether Rolle's Theorem can be applied to $f(x) = (x - 3)(x + 1)^2$ on [-1,3]. Find all values of c such that $f'(c) = 0$.

Rolle's Theorem for Derivatives

Example: Determine whether Rolle's Theorem can be applied to $f(x) = (x - 3)(x + 1)^2$ on [-1,3]. Find all values of c such that $f'(c) = 0$.

$f(-1) = f(3) = 0$ *AND* f is continuous on [-1,3] and diff on (1,3) therefore Rolle's Theorem applies.

$f'(x) = (x-3)(2)(x+1) + (x+1)^2$

$f'(x) = (x+1)(3x-5)$, *set* $= 0$

$c = -1$ *(not* ***interior*** *on the interval) or* $5/3$

$c = 5/3$

Apply Rolle's Theorem

- Apply Rolle's Theorem to the following function $f(x) = x^3 - x$ on [0,1] and compute the location c.

Apply Rolle's Theorem

Apply Rolle's Theorem to the following function f and compute the location c.

$$f(x) = x^3 - x \qquad on \ [0, 1]$$

$$f'(x) = 3x^2 - 1$$

$$f(0) = f(1) = 0$$

By Rolle's Theorem there is a c in [0, 1] *such that*

$$f'(c) = 3c^2 - 1 = 0$$

$$3c^2 - 1 = 0$$

$$3c^2 = 1$$

$$c^2 = \tfrac{1}{3}$$

$$c = \tfrac{1}{\sqrt{3}}, \ [-\tfrac{1}{\sqrt{3}}]$$

The Mean Value Theorem

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$\boxed{1}$ $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$\boxed{2}$ $$f(b) - f(a) = f'(c)(b - a)$$

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The Mean Value Theorem

We can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions.

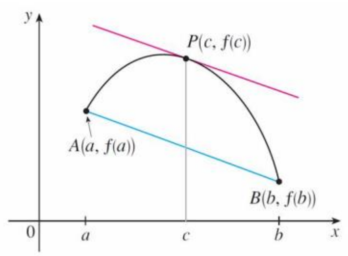

Figure 3

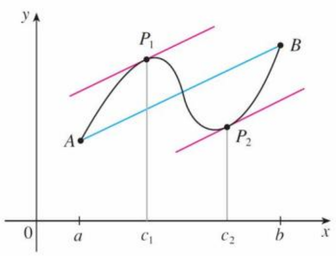

Figure 4

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The Mean Value Theorem

The slope of the secant line *AB* is

$$\boxed{3} \qquad m_{AB} = \frac{f(b) - f(a)}{b - a}$$

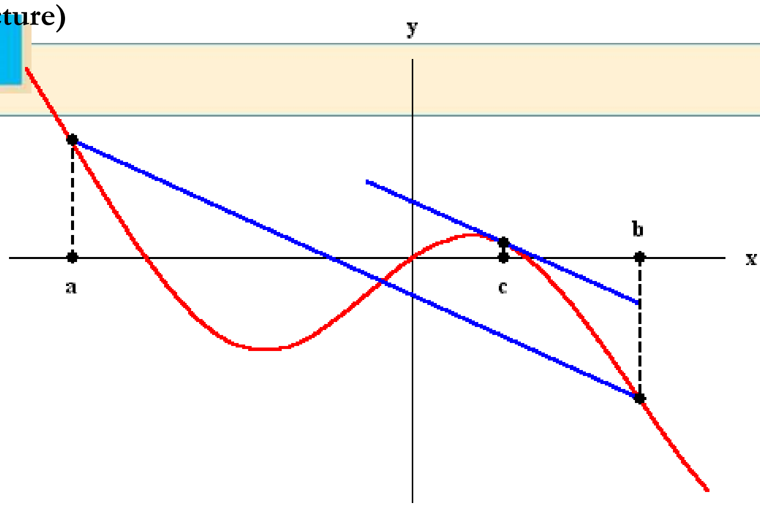
which is the same expression as on the right side of Equation 1.

The Mean Value Theorem

Since $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$, the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line *AB*.

In other words, there is a point *P* where the tangent line is parallel to the secant line *AB*.

(see Picture)



The special case, when $f(a) = f(b)$ is known as Rolle's Theorem. In this case, we have $f'(c) = 0$.

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Example 1

1. Apply the MVT to $f(x) = x^3 - x$ on $[0,2]$.

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Example 1 - Soln

To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, $a = 0$, $b = 2$.

Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on $[0, 2]$ and differentiable on $(0, 2)$.

Therefore, by the Mean Value Theorem, there is a number c in $(0, 2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Example 1 - Soln

Now $f(2) = 6$,

$f(0) = 0$, and

$f'(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2$$

$$= 6c^2 - 2$$

which gives $c^2 = \frac{4}{3}$, that is, $c = \pm 2/\sqrt{3}$. But c must lie in $(0, 2)$, so $c = 2/\sqrt{3}$.

Example 1 - Soln

Figure 6 illustrates this calculation:

The tangent line at this value of *c* is parallel to the secant line *OB*.

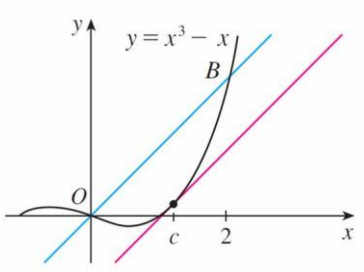

Figure 6

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Example 2

2. Apply the MVT to $f\left(x\right) = -x^2 + 4$ on [-1,4].

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2. Apply the MVT to $f(x) = -x^2 + 4$ on [-1,4].

f(x) is continuous on [-1,4].

MVT applies!

$f'(x) = -2x$

f(x) is differentiable on [-1,4].

$$-2c = \frac{f(4) - f(-1)}{4 - -1}$$

$$-2c = \frac{-15}{5}$$

$$-2c = -3$$

$$c = \frac{3}{2}$$

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Example 3

3. Apply the MVT to $f(x) = x^{2/3}$ on [-1,2].

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3. Apply the MVT to $f(x) = x^{2/3}$ on [-1,2].

$$f'(x) = \tfrac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

f(x) is continuous on [-1,2].

f(x) is not differentiable at x = 0.

MVT does not apply!

Example 4

4. Determine if the mean value theorem applies, and if so find the value of c.

$$f(x) = \frac{x+1}{x} \quad on \ \left[\frac{1}{2}, 2\right]$$

Determine if the mean value theorem applies, and if so find the value of *c*.

$$f(x)=\frac{x+1}{x} \quad on \quad \left[\frac{1}{2},2\right]$$

f is continuous on [1/2, 2], and differentiable on (1/2, 2).

$$\frac{f(2)-f\left(\frac{1}{2}\right)}{2-\frac{1}{2}} =$$

This should equal *f '(x)* at the point c. Now find *f '(x).*

$$f'(x)=\frac{x(1)-(x+1)(1)}{x^2}=$$

Determine if the mean value theorem applies, and if so find the value of *c*.

$$f(x)=\frac{x+1}{x} \quad on \quad \left[\frac{1}{2},2\right]$$

$$\frac{f(2)-f\left(\frac{1}{2}\right)}{2-\frac{1}{2}} = \frac{\frac{3}{2}-3}{\frac{3}{2}} = -1$$

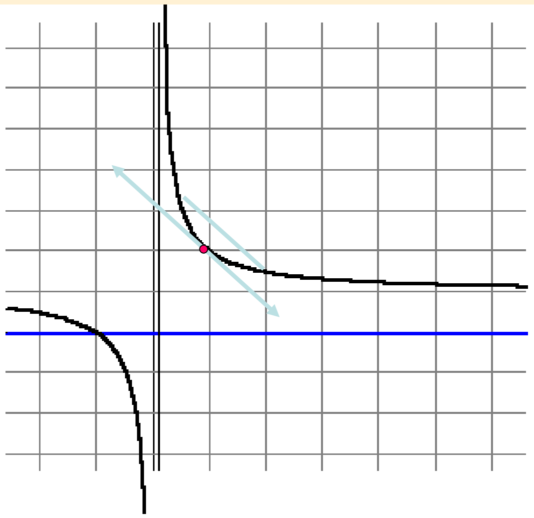
$$f'(x)=\frac{x(1)-(x+1)(1)}{x^2}=-\frac{1}{x^2}$$

$$-\frac{1}{x^2}=-1$$

$$x^2=1$$

$$x=1$$

$$c=1$$



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Alternate form of the Mean Value Theorem for Derivatives

$$\boxed{f(b) = f(a) + (b-a)f'(c)}$$

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Example 5

Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x.
How large can $f(2)$ possibly be?

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Example 5 - Soln

Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x.
How large can $f(2)$ possibly be?

Solution:

We are given that f is differentiable (and therefore continuous) everywhere.

In particular, we can apply the Mean Value Theorem on the interval [0, 2]. There exists a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

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Example 5 – *Solution*

cont'd

so $$f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$$

We are given that $f'(x) \leq 5$ for all x, so in particular we know that $f'(c) \leq 5$.

Multiplying both sides of this inequality by 2, we have $2f'(c) \leq 10$, so

$$f(2) = -3 + 2f'(c) \leq -3 + 10 = 7$$

The largest possible value for $f(2)$ is 7.

Example 6

Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Example 6 – *Solution*

Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Solution:

First we use the Intermediate Value Theorem to show that a root exists. Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$.

Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number c between 0 and 1 such that $f(c) = 0$.

Thus the given equation has a root.

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Example 6 – *Solution*

To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction.

Suppose that it had two roots a and b. Then $f(a) = 0 = f(b)$ and, since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$.

Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$.

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Example 6 – *Solution*

But

$$f'(x) = 3x^2 + 1 \geq 1 \qquad \text{for all } x$$

(since $x^2 \geq 0$) so $f'(x)$ can never be 0. This gives a contradiction.

Therefore the equation can't have two real roots.