Lesson 55 – Numerical Solutions to Differential Equations – Euler's Method

IBHL - Calculus - Santowski

(A) Opening Problem – Newton's Law of Cooling

- Newton's Law of Cooling states that the temperature of a body changes at a rate proportional to the difference in temperature between its own temperature and the temperature of its surroundings
- For example, a ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given (in general) by

$$\frac{dT}{dt} = A\left(T - T_s\right) \quad \text{where } T_S \text{ is the surroundings temp}$$

- Estimate the temperature after 480 seconds

(A) Opening Problem – BUT what if ....

- A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is NOW given (in general) by

$$\frac{dT}{dt} = A\left(T^4 - B\right) \quad \text{where } T(0) = T_0 \text{ (in Kelvin)}$$

- But specifically now by:

$$\frac{dT}{dt} = -2.2067 \times 10^{-12}\left(T^4 - 81 \times 10^8\right) \text{ and } T(0) = 1200K$$

- Estimate the temperature after 480 seconds

Lesson Objectives

- Review solution strategies to solving Diff Eqns ➔ algebraic and graphic (slope fields)
- Find specific numerical solutions to Diff Eqns through successive linear approximations (Euler's method)

Slope Manipulations ….

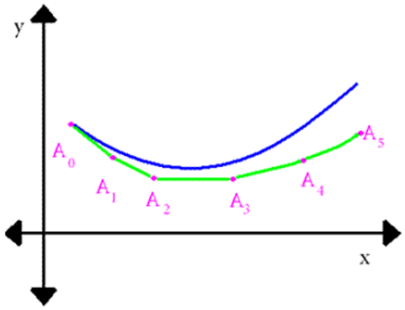
- If $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{\Delta x}$ solve for y_2
- Explain the significance of y_2, given the calculation you have just performed. Explain with a visual ……

(A) Overview

- If we were given a separable differential equation and an initial condition, we could find the solution.
- However, there are some differential equations where this simple technique (separable) will not work ➔ so there are other techniques for solving differential equations, but again they too sometimes fail.

(A) Overview

- This is where Euler's Method is used. Euler's Method provides us with an *approximation* for the solution of a differential equation. The idea behind Euler's Method is to use the concept of *local linearity* to join multiple small line segments so that they make up an approximation of the actual curve, as seen in the diagram.



(A) Overview

- Other times, we may not be interested in a visual solution (graph) & we cannot algebraically develop the equation of the solution, but we may be interested in a SPECIFIC NUMERIC solution ➔ FOR EXAMPLE:
- A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

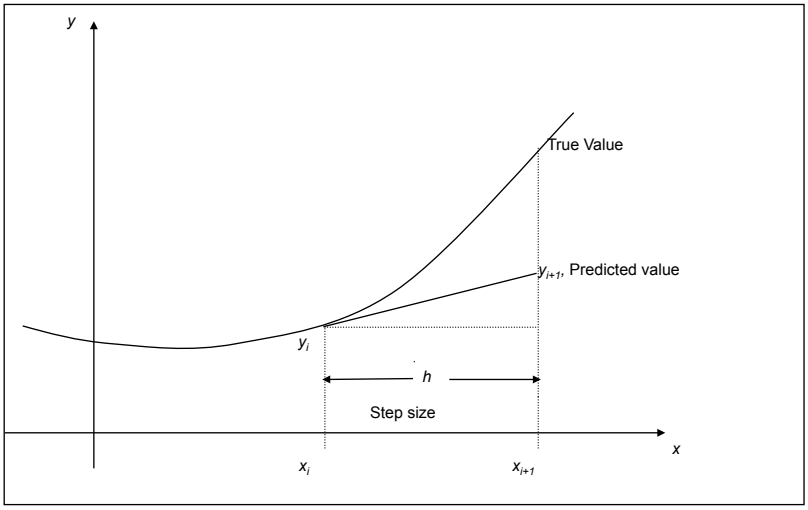
$$\frac{dT}{dt} = -2.2067 \times 10^{-12}\left(T^4 - 81 \times 10^8\right), T(0) = 1200K$$

- and we want an estimate the temperature after 480 seconds

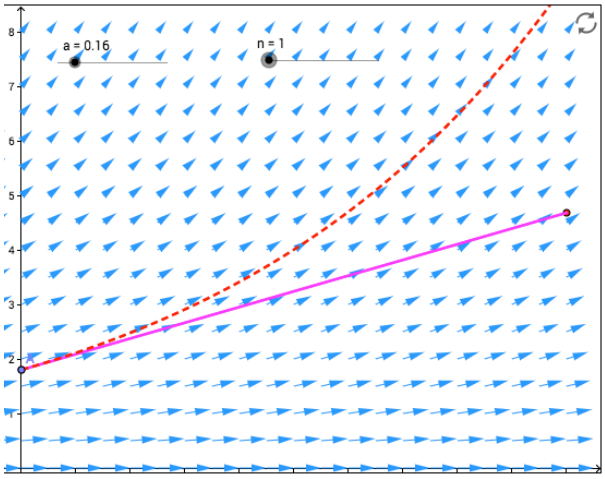
(B) What is Euler's Method?

- METHOD: Essentially, you take an initial condition (a given data point), and use that condition as a starting point to take a small step *h* along the tangent line to get to a new point. Now repeat using the new point as a starting point.
- *Note: The smaller the steps used, the more accurate your graph will be.

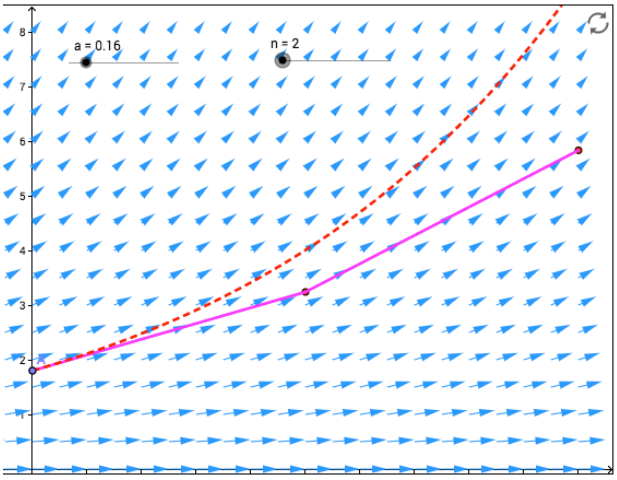
(B) What is Euler's Method?



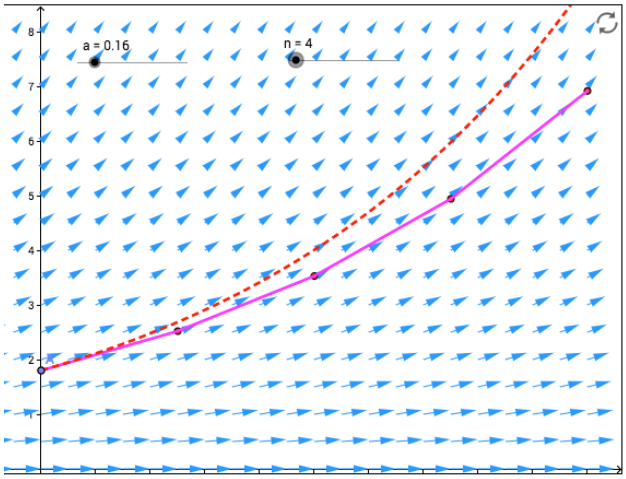
(B) Explain the next 5 slides ….



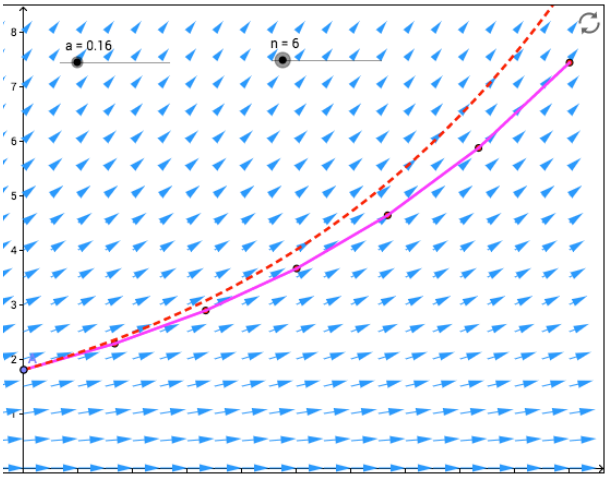
(B) Explain the next 4 slides ….



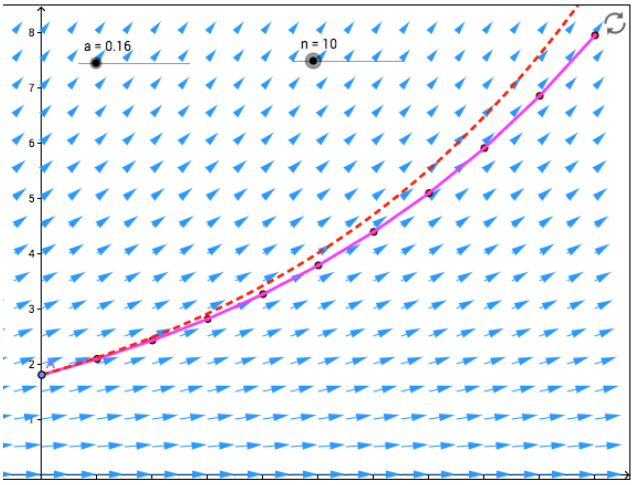
(B) Explain the next 3 slides ….



(B) Explain the next 2 slides ….



(B) Explain the next 1 slide ….



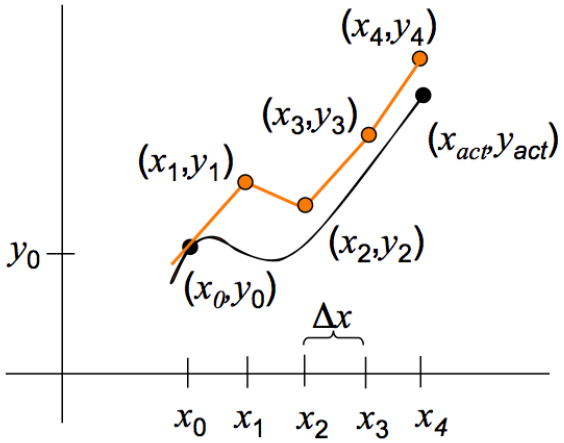
(C) Example

- Given the differential equation

$$\frac{dy}{dx} = x^2 y \text{ and } y(0) = 1$$

find the value of y(2) using Euler's method with 4 iterations.

(C) Example



OVERVIEW: To Apply Euler's Method:

1. Divide the interval n equal subintervals. (In our example 4.)

2. Compute the width of each subinterval which is $\Delta x=h=(x_n-x_0)/n$.

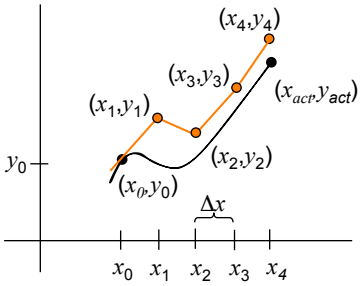
3. Compute the sequence of points as follows:

$$(x_1,y_1)=\begin{cases}x_1=x_0+\Delta x\\ y_1=y_0+f(x_0,y_0)\Delta x\end{cases}$$

$$(x_2,y_2)=\begin{cases}x_2=x_1+\Delta x\\ y_2=y_1+f(x_1,y_1)\Delta x\end{cases}$$

$$(x_3,y_3)=\begin{cases}x_3=x_2+\Delta x\\ y_3=y_2+f(x_2,y_2)\Delta x\end{cases}$$

$$(x_4,y_4)=\begin{cases}x_4=x_3+\Delta x\\ y_4=y_3+f(x_3,y_3)\Delta x\end{cases}$$



In general the coordinates of the point (x_{n+1},y_{n+1}) can be computed from the coordinates of the point (x_n,y_n) as follows:

$$(x_{n+1},y_{n+1})=\begin{cases}x_{n+1}=x_n+\Delta x\\ y_{n+1}=y_n+f(x_n,y_n)\Delta x\end{cases}$$

Example:

Given the differential equation to the right with its boundary conditions **find the value of** *y(2)* using Euler's method with 4 iterations.

$$\frac{dy}{dx} = x^2 y$$
$$y(0) = 1$$

1. $x_0 = 0 \quad and \quad y_0 = 1 \quad and \quad x_4 = 2$

2. $h = \Delta x = \frac{2-0}{4} = \frac{1}{2}$

3. $\begin{cases} x_1 = 0 + \frac{1}{2} = \frac{1}{2} \\ y_1 = 1 + \left(0^2 \cdot 1\right)\frac{1}{2} = 1 \end{cases}$

4. $\begin{cases} x_2 = \frac{1}{2} + \frac{1}{2} = 1 \\ y_2 = 1 + \left(\left(\frac{1}{2}\right)^2 \cdot 1\right)\frac{1}{2} = 1 + \frac{1}{8} = 1.125 \end{cases}$

5. $\begin{cases} x_3 = 1 + \frac{1}{2} = 1.5 \\ y_3 = 1.125 + \left((1)^2 \cdot 1.125\right)\frac{1}{2} = 1.125 + .5625 = 1.6875 \end{cases}$

6. $\begin{cases} x_4 = 1.5 + \frac{1}{2} = 2 \\ y_4 = 1.6875 + \left((1.5)^2 \cdot 1.6875\right)\frac{1}{2} = 1.6875 + 1.89844 = 3.58594 \end{cases}$

The actual solution is:

$$y = e^{\frac{1}{3}x^3}$$

$$y(2) = e^{\frac{8}{3}} = 14.3919$$

Other examples ….

1. Answer the following questions.

   a) Given the differential equation $\frac{dy}{dx} = x + 2$ and $y(0) = 3$. Find an approximation for $y(1)$ by using Euler's method with two equal steps. Sketch you solution.

   b) Solve the differential equation $\frac{dy}{dx} = x + 2$ with the initial condition $y(0) = 3$, and use your solution to find $y(1)$.

   c) The error in using Euler's Method is the difference between the approximate value and the exact value. What was the error in your answer? How could you produce a smaller error using Euler's Method?

Other examples ….

2. Suppose a continuous function f and its derivative f' have values that are given in the following table. Given that $f(2)=5$, use Euler's Method with two steps of size $\Delta x = 0.5$ to approximate the value of $f(3)$.

x	2.0	2.5	3.0
$f'(x)$	0.4	0.6	0.8
$f(x)$	5		

Other examples ….

3. Given the differential equation $\frac{dy}{dx}=\frac{1}{x+2}$ and $y(0)=1$, find an approximation of $y(1)$ using Euler's Method with two steps and step size $\Delta x = 0.5$.

4. Given the differential equation $\frac{dy}{dx}=x+y$ and $y(1)=3$, find an approximation of $y(2)$ using Euler's Method with two equal steps.

Other examples ….

6. Assume that f and f' have the values given in the table. Use Euler's Method with two equal steps to approximate the value of $f(4.4)$.

x	4	4.2	4.4
$f'(x)$	-0.5	-0.3	-0.1
$f(x)$	2		

9. AP 2002-5 (No Calculator)

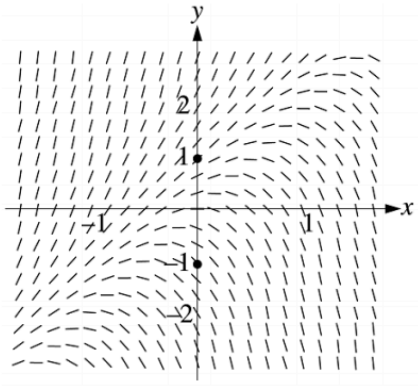
Consider the differential equation: $\frac{dy}{dx} = 2y - 4x$.

a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0,-1)$ and sketch the solution curve that passes through the point $(0,-1)$.

b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

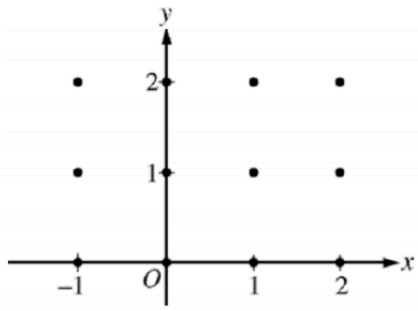
c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0,0)$? If so, is the point a local maximum or a local minimum? Justify your answer.



10. AP 2005-4 (No Calculator)
Consider the differential equation $\frac{dy}{dx} = 2x - y$.

a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated and sketch the solution curve that passes through the point $(0,1)$.



b) The solution curve that passes through the point $(0,1)$ has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the y-coordinate of this local minimum?

c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

d) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.

Other worked examples ….

- http://iss.schoolwires.com/cms/lib4/NC01000579/Centricity/Domain/2872/eulers%20method%20practice2.pdf
- https://mrswood17.wikispaces.com/file/view/6.6+Euler's+worksheet+answers.pdf
- http://liberty.kernhigh.org/wp-content/uploads/2013/09/Eulers-method-2013.pdf
- http://apcentral.collegeboard.com/apc/public/repository/ap09_calculus_bc_q4.pdf

(A) Opening Problem

- A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given (in general) by

$$\frac{dT}{dt} = A\left(T^4 - B\right) \quad \text{where } T(0) = T_0 \text{ (in Kelvin)}$$

- But specifically now by:

$$\frac{dT}{dt} = -2.2067 \times 10^{-12}\left(T^4 - 81 \times 10^8\right) \text{ and } T(0) = 1200K$$

- Estimate the temperature after 480 seconds

(A) Back to Opening Problem

- A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{dT}{dt} = -2.2067 \times 10^{-12}\left(T^4 - 81 \times 10^8\right), T(0) = 1200K$$

- Estimate the temperature after 480 seconds, using h to be 240.

Solution

Step 1:

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + f(t_i,\theta_i)h$$

$$\theta_1 = \theta_0 + f(t_0,\theta_0)h$$

$$= 1200 + f(0,1200)240$$

$$= 1200 + \left(-2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right)\right)240$$

$$= 1200 + (-4.5579)240$$

$$= 106.09K$$

θ_1 is the approximate temperature at $t = t_1 = t_0 + h = 0 + 240 = 240$

$$\theta(240) \approx \theta_1 = 106.09K$$

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Solution Cont

Step 2: For $i = 1, \quad t_1 = 240, \quad \theta_1 = 106.09$

$$\theta_2 = \theta_1 + f(t_1,\theta_1)h$$

$$= 106.09 + f(240,106.09)240$$

$$= 106.09 + \left(-2.2067\times10^{-12}\left(106.09^4 - 81\times10^8\right)\right)240$$

$$= 106.09 + (0.017595)240$$

$$= 110.32K$$

θ_2 is the approximate temperature at $t = t_2 = t_1 + h = 240 + 240 = 480$

$$\theta(480) \approx \theta_2 = 110.32K$$

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Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}\left(0.00333\theta\right)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480)=647.57K$$

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Comparison of Exact and Numerical Solutions

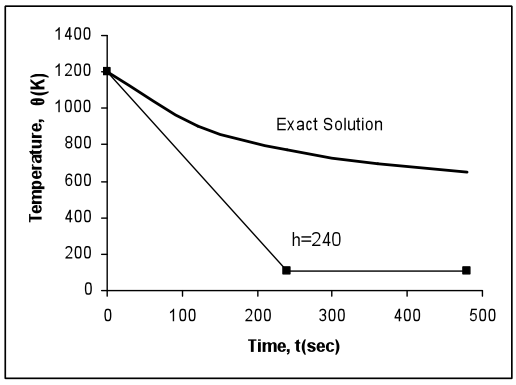

Figure 3. Comparing exact and Euler's method

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Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step, *h*	θ(480)	E_t	$\|\epsilon_t\|$%
480	−987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.77	14.806	2.2864

$$\theta(480) = 647.57K \quad \text{(exact)}$$

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Comparison with exact results

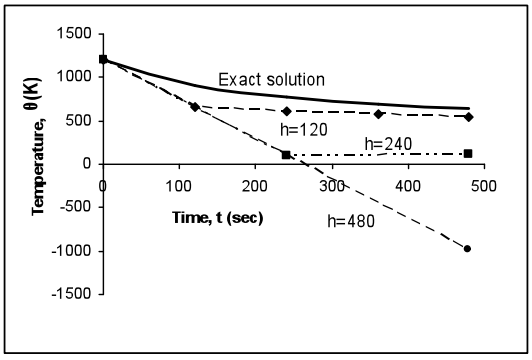

Figure 4. Comparison of Euler's method with exact solution for different step sizes

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Video Resources

- https://www.youtube.com/watch?v=4WGYb87MjCM&index=3&list=PLE11A4E20EA273630
- https://www.youtube.com/watch?v=ty_tLCj3WEk