Lesson 50 – Integration by Parts

Calculus - Santowski

Lesson Objectives

- Use the method of integration by parts to integrate simple power, exponential, and trigonometric functions both in a mathematical context and in a real world problem context

(A) Product Rule

- Recall that we can take the derivative of a product of functions using the product rule:

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

- So we are now going to integrate this equation and see what emerges

(B) Product Rule in Integral Form

- We have the product rule as

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

- And now we will integrate both sides:

$$\int \frac{d}{dx}\left[f(x)\cdot g(x)\right]dx = \int f(x)\cdot g'(x)dx + \int g(x)\cdot f'(x)dx$$
$$f(x)\cdot g(x) = \int f(x)\cdot g'(x)dx + \int g(x)\cdot f'(x)dx$$
rearranging
$$\int f(x)\cdot g'(x)dx = f(x)\cdot g(x) - \int g(x)\cdot f'(x)dx$$

(B) Product Rule in Integral Form

- We will make some substitutions to simplify this equation:

$$\int f(x)\cdot g'(x)dx = f(x)\cdot g(x) - \int g(x)\cdot f'(x)dx$$
let $u = f(x)$ **and let** $v = g(x)$
then $du = f'(x)dx$ **and** $dv = g'(x)dx$
$$\int u\,dv = uv - \int v\,du$$

(C) Integration by Parts Formula

- So we have the formula $\int u\,dv = uv - \int v\,du$
- So what does it mean?
- It seems that if we are trying to solve one integral $\int u\,dv$
- and we create a second integral $\int v\,du$!!!!
- Our HOPE is that the second integral is easier to solve than the original integral

(D) Examples

- Integrate the following functions:

(a) $\int xe^x dx$

(b) $\int x^2 e^x dx$

(c) $\int \ln(x) dx$

(d) $\int x\ln(x) dx$

(D) Examples

- The easiest way to master the method is by practicing, so determine $\int xe^x dx$
- We need to select a *u* and a *dv*

- So we have a choice …..

Let $u = e^x$ and then $dv = xdx$

So $du = e^x dx$ and $v = \frac{1}{2}x^2$

So we get :

$$\int xe^x dx = e^x \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 e^x dx$$

- Comment: Is the second integral any easier than the original???

(D) Examples

- So let's make the other choice as we determine $\int xe^x dx$

Let $u = x$ and then $dv = e^x dx$

So $du = dx$ and $v = e^x$

So we get :

$$\int xe^x dx = xe^x - \int e^x dx$$

- Checkpoint: Is our second integral any "easier" than our first one???

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

- Now verify by differentiating the answer

Example b:

$$\int u\, dv = uv - \int v\, du$$

$$\int x^2 e^x\, dx$$

$u = x^2$	$dv = e^x dx$
$du = 2x\, dx$	$v = e^x$

$$u\, v - \int v\, du$$

$$x^2 e^x - \int e^x \cdot 2x\, dx$$

$$x^2 e^x - 2\int xe^x\, dx$$

$u = x$	$dv = e^x dx$
$du = dx$	$v = e^x$

$$x^2 e^x - 2\left(xe^x - \int e^x dx\right)$$

$$x^2 e^x - 2xe^x + 2e^x + C$$

Example c:

$$\int u\, dv = uv - \int v\, du$$

$$\int \ln x\, dx$$

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}dx \qquad v = x$$

$$u\, v - \int v\, du$$

$$\ln x \cdot x - \int x \cdot \frac{1}{x}\, dx$$

$$x\ln x - x + C$$

(D) More Examples

- Integrate the following functions:

(a) $\int x\cos(x) dx$

(b) $\int e^x \cos(x) dx$

(c) $\int x^2 \cos(x) dx$

(d) $\int \arcsin(x) dx$

Example a:

$$\int u\ dv = uv - \int v\ du$$

$$\int x \cdot \cos x\ dx$$

polynomial factor

$u = x \qquad dv = \cos x\ dx$

$du = dx \qquad v = \sin x$

$$u\ v - \int v\ du$$

$$x \cdot \sin x - \int \sin x\ dx$$

$$x \cdot \sin x + \cos x + C$$

Example b:

$$\int e^x \cos x\ dx$$

$u = e^x \qquad dv = \cos x\ dx$

$du = e^x\ dx \qquad v = \sin x$

$$u\ v - \int v\ du$$

$u = e^x \qquad dv = \sin x\ dx$

$du = e^x\ dx \qquad v = -\cos x$

$$e^x \sin x - \int \sin x \cdot e^x dx$$

$$e^x \sin x - \Big(\underbrace{e^x \cdot -\cos x}_{uv} - \int \underbrace{-\cos x \cdot e^x\ dx}_{v\ du}\Big)$$

$$e^x \sin x + e^x \cos x - \int e^x \cos x\ dx$$

This is the expression we started with!

Example b: CONTINUED)

$$\int e^x \cos x\ dx$$

$u = e^x \qquad dv = \cos x\ dx$

$du = e^x\ dx \qquad v = \sin x$

$$u\ v - \int v\ du$$

$u = e^x \qquad dv = \sin x\ dx$

$du = e^x\ dx \qquad v = -\cos x$

$$e^x \sin x - \int \sin x \cdot e^x dx$$

$$e^x \sin x - \left(e^x \cdot -\cos x - \int -\cos x \cdot e^x\ dx\right)$$

$$\int e^x \cos x\ dx = e^x \sin x + e^x \cos x - \int e^x \cos x\ dx$$

$$2\int e^x \cos x\ dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x\ dx = \frac{e^x \sin x + e^x \cos x}{2} + C$$

Example b:

$$\int e^x \cos x\ dx$$

This is called "solving for the unknown integral."

It works when both factors integrate and differentiate forever.

$$u\ v - \int v\ du$$

$$e^x \sin x - \int \sin x \cdot e^x dx$$

$$e^x \sin x - \left(e^x \cdot -\cos x - \int -\cos x \cdot e^x\ dx\right)$$

$$\int e^x \cos x\ dx = e^x \sin x + e^x \cos x - \int e^x \cos x\ dx$$

$$2\int e^x \cos x\ dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x\ dx = \frac{e^x \sin x + e^x \cos x}{2} + C$$

(E) Further Examples

- For the following question, do it using the method requested. Reconcile your solution(s)

$\int x\sqrt{x+1}\,dx$ by Parts

$\int x\sqrt{x+1}\,dx$ by Substitution

(E) Further Examples

- Definite Integrals ➔ Evaluate:

$$(a)\ \int_1^e \frac{\ln x}{x^2} dx$$

$$(b)\ \int_0^1 \tan^{-1}(x) dx$$

- Calculate $\int_0^1 \tan^{-1} x\,dx$
  - Let $u = \tan^{-1} x$ $\quad dv = dx$
  - Then, $du = \dfrac{dx}{1+x^2}$ $\quad v = x$

$$\int_0^1 \tan^{-1} x\,dx = x\tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$
$$= 1\cdot\tan^{-1}1 - 0\cdot\tan^{-1}0 - \int_0^1 \frac{x}{1+x^2}\,dx$$
$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx$$

- To evaluate this integral, we use the substitution $t = 1 + x^2$ (since u has another meaning in this example).
  - Then, $dt = 2x\,dx$.
  - So, $x\,dx = ½\,dt$.

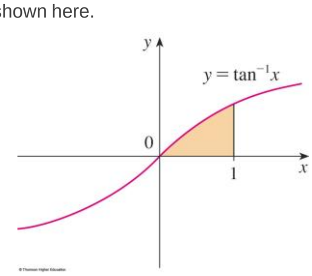
- When $x = 0$, $t = 1$, and when $x = 1$, $t = 2$.
- Hence, $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\int_1^2 \frac{dt}{t}$

$$= \tfrac{1}{2}\ln|t|\Big]_1^2$$
$$= \tfrac{1}{2}(\ln 2 - \ln 1)$$
$$= \tfrac{1}{2}\ln 2$$

- Therefore,

$$\int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx$$
$$= \frac{\pi}{4} - \frac{\ln 2}{2}$$

- As $\tan^{-1}x \geq$ for $x \geq 0$, the integral in the example can be interpreted as the area of the region shown here.



Further Examples

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$
2. $\int \theta \cos \pi\theta\, d\theta$
3. $\int t^2 \cos t\, dt$
4. $\int x^2 \sin x\, dx$
5. $\int_1^2 x \ln x\, dx$
6. $\int_1^e x^3 \ln x\, dx$
7. $\int x e^x\, dx$
8. $\int x e^{3x}\, dx$
9. $\int x^2 e^{-x}\, dx$
10. $\int (x^2 - 2x + 1) e^{2x}\, dx$
11. $\int \tan^{-1} y\, dy$
12. $\int \sin^{-1} y\, dy$
13. $\int x \sec^2 x\, dx$
14. $\int 4x \sec^2 2x\, dx$
15. $\int x^3 e^x\, dx$
16. $\int p^4 e^{-p}\, dp$
17. $\int (x^2 - 5x) e^x\, dx$
18. $\int (r^2 + r + 1) e^r\, dr$
19. $\int x^5 e^x\, dx$
20. $\int t^2 e^{4t}\, dt$
21. $\int e^\theta \sin \theta\, d\theta$
22. $\int e^{-y} \cos y\, dy$
23. $\int e^{2x} \cos 3x\, dx$
24. $\int e^{-2x} \sin 2x\, dx$

Using Substitution

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\, ds$
26. $\int_0^1 x\sqrt{1-x}\, dx$

Further Examples

Using Substitution

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\, ds$
26. $\int_0^1 x\sqrt{1-x}\, dx$
27. $\int_0^{\pi/3} x \tan^2 x\, dx$
28. $\int \ln (x + x^2)\, dx$
29. $\int \sin (\ln x)\, dx$
30. $\int z (\ln z)^2\, dz$

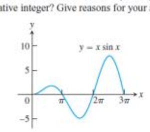
Evaluating Integrals

Evaluate the integrals in Exercises 31–50. Some integrals do not require integration by parts.

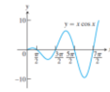
31. $\int x \sec x^2\, dx$
32. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx$
33. $\int x (\ln x)^2\, dx$
34. $\int \frac{1}{x(\ln x)^2}\, dx$
35. $\int \frac{\ln x}{x^2}\, dx$
36. $\int \frac{(\ln x)^3}{x}\, dx$
37. $\int x^3 e^{x^4}\, dx$
38. $\int x^5 e^{x^3}\, dx$
39. $\int x^3 \sqrt{x^2 + 1}\, dx$
40. $\int x^2 \sin x^3\, dx$
41. $\int \sin 3x \cos 2x\, dx$
42. $\int \sin 2x \cos 4x\, dx$
43. $\int e^x \sin e^x\, dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$
45. $\int \cos \sqrt{x}\, dx$
46. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$
47. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$
48. $\int_0^{\pi/2} x^3 \cos 2x\, dx$
49. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt$
50. $\int_0^{1/\sqrt{2}} 2x \sin^{-1} (x^2)\, dx$

Theory and Examples

51. **Finding area** Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis (see the accompanying figure) for
    a. $0 \le x \le \pi$.
    b. $\pi \le x \le 2\pi$.
    c. $2\pi \le x \le 3\pi$.
    d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



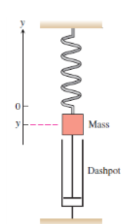
52. **Finding area** Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the accompanying figure) for
    a. $\pi/2 \le x \le 3\pi/2$.
    b. $3\pi/2 \le x \le 5\pi/2$.
    c. $5\pi/2 \le x \le 7\pi/2$.
    d. What pattern do you see? What is the area between the curve and the x-axis for
$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$
n an arbitrary positive integer? Give reasons for your answer.



53. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.
54. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$
    a. about the y-axis.
    b. about the line $x = 1$.
55. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about
    a. the y-axis.
    b. the line $x = \pi/2$.
56. **Finding volume** Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about
    a. the y-axis.
    b. the line $x = \pi$.

    (See Exercise 51 for a graph.)

57. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.
    a. Find the area of the region.
    b. Find the volume of the solid formed by revolving this region about the x-axis.
    c. Find the volume of the solid formed by revolving this region about the line $x = -2$.
    d. Find the centroid of the region.
58. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$, and $x = 1$.
    a. Find the area of the region.
    b. Find the volume of the solid formed by revolving this region about the y-axis.
59. **Average value** A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is
$$y = 2e^{-t} \cos t, \qquad t \ge 0.$$
Find the average value of y over the interval $0 \le t \le 2\pi$.



Test Qs

(a) (5 points). $\int x \tan^{-1} x \, dx$

Q.2. (5 − points). Evaluate $\int \ln(2x)\, dx$.

(a) (5 points). $\int x \tan^{-1} x \, dx$

Solution.

Let $u = \tan^{-1} x$,

then $du = \frac{dx}{1+x^2}$

$dv = x dx$, $v = \frac{x^2}{2}$

..................................(2 points)

Hence $\int x \tan^{-1} x \, dx$

$= \int u dv = uv - \int v dx$

$= \frac{x^2}{2} \tan^{-1} x - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} dx$.....(1 point)

$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$

$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \left[\int \left(1 - \frac{1}{1+x^2}\right) dx \right]$

$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C$.....(2 points)

Q.2. (5 − points). Evaluate $\int \ln(2x)\, dx$.

SOLUTION.

$$\left.\begin{array}{ll} u = \ln(2x) & dv = dx \\ du = \frac{2dx}{2x} = \frac{dx}{x} & v = x \end{array}\right\} \ldots\ldots \longrightarrow (2 - points)$$

$$\int \ln(2x)\, dx = x \ln(2x) - \int dx \ \ldots\ldots \longrightarrow (2 - points)$$

$$= x \ln(2x) - x + C \ \ldots\ldots\ldots \longrightarrow (1 - point)$$

INTEGRATION BY PARTS

- Evaluate ∫ e^x sin*x dx*
  - e^x does not become simpler when differentiated.
  - Neither does sin *x* become simpler.

INTEGRATION BY PARTS

- Nevertheless, we try choosing
  u = e^x and *dv* = sin *x*
  - Then, *du* = e^x *dx* and *v* = – cos *x*.

INTEGRATION BY PARTS

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

- So, integration by parts gives:

The integral we have obtained, ∫e^xcos *xdx*, is no simpler than the original one.

At least, it's no more difficult.

Having had success in the preceding example integrating by parts twice, we do it again.

INTEGRATION BY PARTS

- This time, we use $u = e^x$ and $dv = \cos x\, dx$
- Then, $du = e^x\, dx$, $v = \sin x$, and

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$$

INTEGRATION BY PARTS

- At first glance, it appears as if we have accomplished nothing.
  - We have arrived at $\int e^x \sin x\, dx$, which is where we started.

INTEGRATION BY PARTS

- However, if we put the expression for $\int e^x \cos x\, dx$ from Equation 5 into Equation 4, we get:

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$$

  - This can be regarded as an equation to be solved for the unknown integral.

INTEGRATION BY PARTS

- Adding to both sides $\int e^x \sin x\, dx$, we obtain:

$$2\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x$$

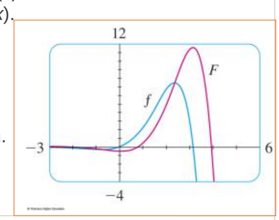
INTEGRATION BY PARTS

- Dividing by 2 and adding the constant of integration, we get:

$$\int e^x \sin x\, dx = \tfrac{1}{2} e^x (\sin x - \cos x) + C$$

INTEGRATION BY PARTS

- The figure illustrates the example by showing the graphs of $f(x) = e^x \sin x$ and $F(x) = ½\, e^x(\sin x - \cos x)$.
  - As a visual check on our work, notice that $f(x) = 0$ when F has a maximum or minimum.



(F) Internet Links

- Integration by Parts from Paul Dawkins, Lamar University
- Integration by Parts from Visual Calculus