Lesson 47 – Integration by Substitution

IBHL - Calculus - Santowski

Lesson Objectives

- Use the method of substitution to integrate simple composite power, exponential, logarithmic and trigonometric functions both in a mathematical context and in a real world problem context

Fast Five

- Differentiate the following functions:

$$\frac{d}{dx}\left(x^2+5\right)^3$$

$$\frac{d}{dx}e^{x^2}$$

$$\frac{d}{dx}\sin\left(\frac{4}{x^3}\right)$$

$$\frac{d}{dx}\ln\left(x^3+1\right)$$

$$\frac{d}{dx}\sin\left(\ln\left(\sqrt{x+3}\right)\right)$$

$$\frac{d}{dx}\frac{1}{\left(x^2+6x\right)^2}$$

(A) Introduction

- At this point, we know how to do simple integrals wherein we simply apply our standard integral "formulas"
- But, similar to our investigation into differential calculus, functions become more difficult/challenging, so we developed new "rules" for working with more complex functions
- Likewise, we will see the same idea in integral calculus and we shall introduce 2 methods that will help us to work with integrals

(B) "Simple" Examples ????

- Find the following:

$$\int \sqrt[4]{x}\, dx$$
$$\int \frac{1}{t^3}\, dt$$
$$\int \cos w\, dw$$
$$\int e^y\, dy$$

- Now, try these:

$$\int 2xe^{x^2}\, dx$$
$$\int \frac{1}{x^4} \sin\left(\frac{4}{x^3}\right) dx$$

(C) Looking for Patterns

- Alright, let's use wolframalpha to help us with some of the following integrals:
- Now, look at our fast 5
- Now, let's look for patterns???
- Examples

$$\int 2xe^{x^2}\, dx$$
$$\int -\frac{12}{x^4} \sin\left(\frac{4}{x^3}\right) dx$$
$$\int \frac{3x^2}{x^3+1}\, dx$$
$$\int \frac{\cos(\ln(x))}{x}\, dx$$

(C) Looking for Patterns

- So, in all the integrals presented here, we see that some part of the function to be integrated is a COMPOSED function and then the second pattern we observe is that we also see some of the derivative of the "inner" function appearing in the integral
- Here are more examples to illustrate our "pattern"

$$\int 9x^2 \sqrt[4]{6x^3+5}\, dx$$
$$\int \left(1-\frac{1}{w}\right)\cos(w-\ln w)dw$$
$$\int (16y-2)e^{4y^2-y} dy$$

(D) Generalization from our Pattern

- So we can make the following generalization from our observation of patterns:

$$\int f(g(x))\cdot g'(x)\, dx = \int f(u)\, du$$
$$\text{where } u = g(x) \text{ and then } du = g'(x)dx$$

- But the question becomes: how do we know what substitution to make???
- Generalization: ask yourself what portion of the integrand has an inside function and can you do the integral with that inside function present. If you can't then there is a pretty good chance that the inside function will be the substitution.

(E) Working out Some Examples

In these problems, a substitution is given.

1. $\int (3x-5)^{17}\,dx,\ u = 3x-5$
2. $\int_0^4 x\sqrt{x^2+9}\,dx,\ u = x^2+9$
3. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx,\ u = \sqrt{x}.$
4. $\int \frac{\cos 3x\,dx}{5+2\sin 3x},\ u = 5+2\sin 3x$

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(E) Working out Some Examples

- Integrate $\int x^2\left(3-10x^3\right)^4 dx$
- Integrate $\int 9x^2\sqrt[4]{6x^3+5}\,dx$

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(E) Working out Some Examples

- Integrate $\int x^2\left(3-10x^3\right)^4 dx$

Let $u = 3-10x^3$ then $du = -30x^2dx$

so $x^2dx = -\frac{1}{30}du$ and we get :

$$= \int u^4 \cdot -\frac{1}{30}du = -\frac{1}{30}\int u^4 du$$

$$= -\frac{1}{30}\cdot\frac{u^{4+1}}{4+1} + C$$

$$= -\frac{1}{150}\left(3-10x^3\right)^5 + C$$

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(E) Working out Some Examples

- Integrate $\int 9x^2\sqrt[4]{6x^3+5}\,dx$

Let $u = 6x^3+5$ then $du = 18x^2dx$

so $9x^2dx = \frac{1}{2}du$ and we get :

$$= \int \sqrt[4]{u}\cdot\frac{1}{2}du = \frac{1}{2}\int u^{1/4}du$$

$$= \frac{1}{2}\cdot\frac{u^{1/4+1}}{1/4+1} + C$$

$$= \frac{2}{5}\left(6x^3+5\right)^{\frac{5}{4}} + C$$

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(E) Working out Some Examples

In these problems, you need to determine the substitution yourself.

5. $\int (4-3x)^7\,dx.$

6. $\int_{\pi/4}^{\pi/3} \csc^2(5x)\,dx$

7. $\int x^2 e^{3x^3-1}\,dx$

(F) Further Examples

- Integrate the following:

$\int x^2 e^{x^3}\,dx$

$\int \frac{x^2}{\sqrt{1-x^3}}dx$

$\int \frac{\ln x}{x}dx$

$\int \sin^4(x)\cos(x)dx$

$\int \sin x \cos x dx$

$\int \tan x dx$

$\int \cos(3x)\sin^{10}(3x)dx$

$\int x^2 \sin(x^3)dx$

(F) Further Examples

- Integrate the following:

$\int_0^{\pi/4} \tan x \sec^2 x\,dx$

$\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

$\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx$

(F) Challenge Examples

- Integrate the following:

(a) $\int \frac{x}{x+1}\,dx$

(b) $\int \sin^2(x)\,dx$

(c) $\int \cos^2(x)\,dx$

(d) $\int \sec(x)\,dx$

(F) Challenge Examples

Sometimes there is more than one way to skin a cat:

8. Find $\int \frac{x}{1+x}\,dx$, both by long division and by substituting $u = 1 + x$.

9. Find $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$, both by substituting $u = z^2 + 1$ and $u = \sqrt[3]{z^2+1}$.

(F) Challenge Examples

- Integrate the following:

(a) $\int \frac{x^2}{x+1}\,dx$

(b) $\int \frac{x^4+x-4}{x^2-2}\,dx$ (c) $\int \frac{x^5-35x}{x^2+6}\,dx$

(d) $\int \frac{dx}{x^2-4x+4}$ (e) $\int \frac{dx}{\sqrt{-x^2+4x-3}}$

(f) $\int \frac{dx}{2+9x^2}$ (g) $\int \frac{dx}{\sqrt{4-25x^2}}$

Working with Trig Substitutions

- Given the following expressions, simplify the expression, given the suggested substitution:

(a) Simplify $\sqrt{a^2 - x^2}$ given that $x = a\sin\theta$

(b) Simplify $\sqrt{a^2 + x^2}$ given that $x = a\tan\theta$

(c) Simplify $\sqrt{x^2 - a^2}$ given that $x = a\sec\theta$

Integration by Trig Substitutions

- Use the suggested trig substitutions to find the following:

(a) $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$ using $x = 3\sin\theta$

(b) $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ using $x = 2\tan\theta$

(c) $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$ using $x+1=u$ and then .....?

Further Substitutions

- Given the ellipse $4x^2 + y^2 = 4$, determine:

- (a) the x-intercepts
- (b) the area between the ellipse, the x-axis and the zeroes

Further Substitutions

- Integrate the following indefinite integrals:

$$(a)\ \int \sqrt{16 - x^2}\, dx$$

$$(b)\ \int \sqrt{x^2 - 16}\, dx$$

$$(c)\ \int \frac{\sqrt{9 - x^2}}{x^2}\, dx$$

$$(d)\ \int \frac{dx}{x^2\sqrt{x^2 + 4}}$$

$$(e)\ \int \frac{x}{\sqrt{x^2 + 4}}\, dx$$

CHALLENGE

- Evaluate:

$$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\left(4x^2 + 9\right)^{3/2}}\, dx$$

- ANS = 3/32