Lesson 46 – Working with Definite Integrals

HL Math- Santowski

Lesson Objectives

- 1. Calculate simple definite integrals
- 2. Calculate definite integrals using the properties of definite integrals
- 3. Determine total areas under curves
- 4. Apply definite integrals to a real world problems

(A) Review

- But we have introduced a new symbol for our antiderivative → the integral symbol (∫)
- So now we can put our relationships together

$$A(x) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

- Where $\int_a^b f(x)dx$ is now referred to as a definite integral

(B) The Fundamental Theorem of Calculus

- The fundamental theorem of calculus shows the connection between antiderivatives and definite integrals
- Let f be a continuous function on the interval $[a,b]$ and let F be any antiderivative of f. Then $\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$

(C) Examples

- Let's work with the FTC on the following examples:

$$(i)\ \int_2^5 \left(5x^2 - 4x + 5\right)dx$$

$$(ii)\ \int_1^2 \frac{dy}{y}$$

$$(iii)\ \int_0^{\pi} \sin x\,dx$$

$$(iv)\ \int_0^{\frac{\pi}{2}} \left(2e^{2x} - 5\cos x\right)dx$$

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(C) Examples

- Evaluate the following integrals:

$$(a)\ \int_0^{\frac{\pi}{3}} \left(\frac{2}{\pi}x - 2\sec^2 x\right)dx$$

$$(b)\ \int_0^{\frac{\pi}{2}} (\cos\theta + 2\sin\theta)\,d\theta$$

$$(c)\ \int_{-e}^{-e^2} \frac{3}{x}dx$$

$$(d)\ \int_{\ln 6}^{\ln 3} 8e^t\,dt$$

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(D) Properties of Definite Integrals

- To further develop definite integrals and there applications, we need to establish some basic properties of definite integrals:

$$(i)\ \int_a^a f(x)dx = 0$$

$$(ii)\ \int_a^b (k \times f(x))dx = k \times \int_a^b f(x)dx$$

$$(iii)\ \int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$(iv)\ \int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$(v)\ \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

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(E) Further Examples

- Let's work with the FTC on the following examples.

$$(i)\ \int_0^4 \left(x^2 - 4\right)dx$$

$$(ii)\ \int_0^4 \left(x^2 - 3x\right)dx$$

$$(iii)\ \int_0^4 \left|x^2 - 4\right|dx$$

(*iv*) Find the TOTAL between the x-axis and the function $y = x^2 - 3x$

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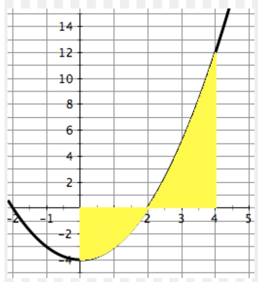
(E) Further Examples

- Let's work with the FTC on the following examples

$$(i)\ \int_0^2 (x^2-4)dx$$
$$(ii)\ \int_1^3 (x^2-3x)dx$$
$$(iii)\ \int_0^4 (x^2-4)dx$$



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(E) Further Examples

Evaluate and interpret the definite integral in terms of areas.

$$\int_0^4 \left(x^3-7x^2+11x\right)dx$$

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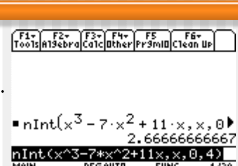
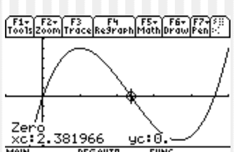
(E) Further Examples

Interpret the definite integral in terms of areas.

$$\int_0^4 \left(x^3-7x^2+11x\right)dx$$





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(E) Further Examples

- So the answer to the question of interpreting the definite integral $\int_0^4 \left(x^3-7x^2+11x\right)dx$ in terms of areas is:
- (i) 2.67 if the interpretation is as the difference between the positive area and the negative area (NET area)
- (ii) 12.77 if we interpret the answer to mean the TOTAL area of the regions

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(E) Further Examples

- We are going to set up a convention here in that we would like the area to be interpreted as a positive number when we are working on TOTAL Area interpretations of the DI
- So, we will take the absolute value of negative "areas" or values
- Then the second consideration will be when a function has an x-intercept in the interval $[a,b] \rightarrow$ we will then break the area into 2 (or more) sub-intervals $[a,c]$ & $[c,b]$ and work with 2 separate definite integrals

(F) DEFINITE INTEGRALS & DISCONTINUOUS FUNCTIONS

- Evaluate the following definite integrals:

$$(a)\ \int_2^5 |x-3|\,dx$$

$$(b)\ \int_{-2}^2 |x^2-1|\,dx$$

$$(c)\ \int_{-2}^4 f(x)dx \text{ where } f(x)=\begin{cases} 2+x^2 & x<0 \\ \frac{1}{2}x+2 & x\ge 0 \end{cases}$$

$$(d)\ \int_0^{\frac{3\pi}{2}} f(x)dx \text{ where } f(x)=\begin{cases} 2\sin x & x\le \frac{\pi}{2} \\ 2+\cos x & x>\frac{\pi}{2} \end{cases}$$

(F) DEFINITE INTEGRALS & DISCONTINUOUS FUNCTIONS

- True or false & explain your answer:

$$\int_0^{2\pi} \sec^2 x\,dx = \tan x\Big|_0^{2\pi} = 0-0=0$$

Examples #1

- Find the area between the x-axis and the curve $f(x)=x^2+1$ on the interval $-1<x<2$

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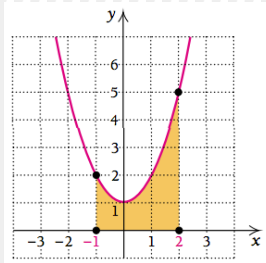


Examples #1

- Find the area between the x-axis and the curve $f(x)=x^2+1$ on the interval $-1 < x < 2$

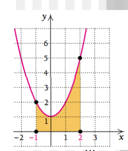
Solution In this case, $f(x) = x^2 + 1$, $a = -1$, and $b = 2$.

1) Find any antiderivative $F(x)$ of $f(x)$. We choose the simplest one:

$$F(x) = \frac{x^3}{3} + x.$$

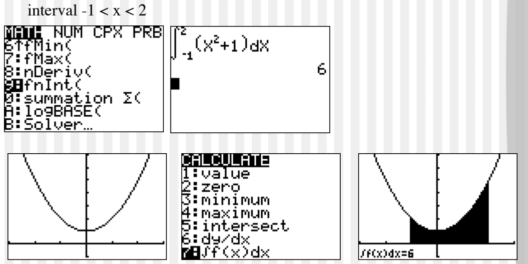
2) Substitute 2 and -1 and find the difference $F(2) - F(-1)$:

$$\begin{aligned} F(2) - F(-1) &= \left[\frac{2^3}{3} + 2\right] - \left[\frac{(-1)^3}{3} + (-1)\right] \\ &= \frac{8}{3} + 2 - \left[\frac{-1}{3} - 1\right] \\ &= \frac{8}{3} + 2 + \frac{1}{3} + 1 \\ &= 6. \end{aligned}$$



Examples #1 – Using TI-84

- Find the area between the x-axis and the curve $f(x)=x^2+1$ on the interval $-1 < x < 2$



Examples #2

- Find the area under the curves specified below:

(a) $f(x) = \dfrac{1}{x+3}$ on $\left[-2,1\right]$

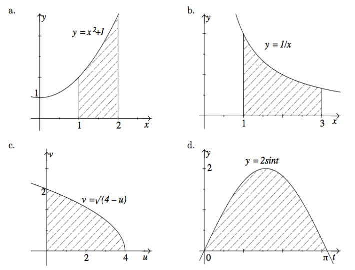
(b) $g(x) = 1 + \sin 2x$ on $\left[0,\dfrac{\pi}{4}\right]$

(c) $h(x) = 4 - x + x^3$ on $\left[-1,2\right]$

(d) $m(x) = e^x - x$ on $\left[0,e\right]$

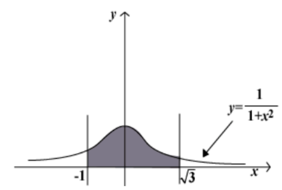
Examples #3

3. Find the area of the shaded region in each of the diagrams below:



Examples #4

Find the area bounded by $y=\frac{1}{1+x^2}$, $x=-1$, $x=\sqrt{3}$ and the x-axis i.e. find the area of shaded region R in the picture below:



Examples #4

Find the area bounded by $y=\frac{1}{1+x^2}$, $x=-1$, $x=\sqrt{3}$ and the x-axis i.e. find the area of shaded region R in the picture below:

$$\text{Area} = \int_{-1}^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \left[\text{Arctan}x\right]_{-1}^{\sqrt{3}}$$

$$= \text{Arctan}\sqrt{3} - \text{Arctan}(-1)$$

$$= \frac{\pi}{3} - \left(-\frac{\pi}{4}\right)$$

$$= \frac{7\pi}{12}.$$

Example #5

- For the following definite integrals:

- (i) Evaluate without the TI-84
- (ii) Evaluate using a graph on the TI-84
- (iii) Interpret your results

(a) $\int_0^2 -x^2 dx$

(b) $\int_{-1}^2 \left(x^2 - 1\right) dx$

Example #6

Evaluate: $\int_{-1}^{2} (x^3 - 3x + 1)\, dx$. Interpret the results in terms of area.

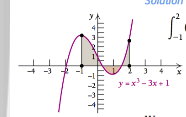
Example

Find the area between the curve $y = x(x-3)$ and the ordinates $x = 0$ and $x = 5$.

Solution

Evaluate: $\int_{-1}^{2} (x^3 - 3x + 1)\, dx$. Interpret the results in terms of area.

EXAMPLE 11 Evaluate: $\int_{-1}^{2} (x^3 - 3x + 1)\, dx$. Interpret the results in terms of area.



Solution We have

$$\int_{-1}^{2} (x^3 - 3x + 1)\, dx = \left[\frac{x^4}{4} - \frac{3}{2}x^2 + x\right]_{-1}^{2}$$

$$= \left(\frac{2^4}{4} - \frac{3}{2}\cdot 2^2 + 2\right) - \left(\frac{(-1)^4}{4} - \frac{3}{2}\cdot(-1)^2 + (-1)\right)$$

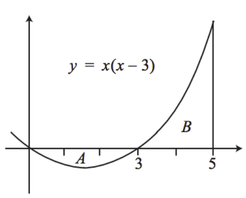
$$= \left(\frac{16}{4} - 6 + 2\right) - \left(\frac{1}{4} - \frac{3}{2} - 1\right) = 0 - \left(-2\tfrac{1}{4}\right) = 2\tfrac{1}{4}.$$

We can graph the function $f(x) = x^3 - 3x + 1$ over the interval and shade the area between the curve and the x-axis. The sum of the areas above the axis minus the area below is $2\frac{1}{4}$.

HINT:

Example

Find the area between the curve $y = x(x-3)$ and the ordinates $x = 0$ and $x = 5$.



Example #7

2. Find the area contained by the curve $y = x(x-1)(x+1)$ and the x-axis.

3. Calculate the value of

$$\int_{-1}^{1} x(x-1)(x+1)\,\mathrm{d}x\,.$$

Compare your answer with that obtained in question 3, and explain what has happened.

4. Calculate the value of

$$\int_{0}^{6} (4x - x^2)\,\mathrm{d}x\,.$$

Explain your answer.

Example #8

1. Find the area enclosed by the graph of $y = 3x^2(x - 4)$ and the x-axis.

2. i Find the value of $\int_0^{2\pi} \sin x dx$.

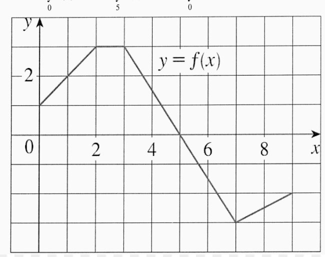
   ii Find the area enclosed between the graph of $y = \sin x$ and the x-axis from $x = 0$ to $x = 2\pi$.

3. Find the total area enclosed between the graph of $y = 12x(x + 1)(2 - x)$ and the x-axis.

Example #9

The graph of f is shown below. Evaluate each integral by interpreting it in terms of areas.

a) $\int_0^2 f(x)dx$ b) $\int_0^5 f(x)dx$ c) $\int_5^7 f(x)dx$ d) $\int_0^9 f(x)dx$



Example #10

13. If $\int_1^5 f(x)dx = 12$ and $\int_4^5 f(x)dx = 3.6$, find $\int_1^4 2f(x)dx$

14. If $\int_0^9 f(x)dx = 37$ and $\int_0^9 g(x)dx = 16$, find $\int_0^9 [2f(x) + 3g(x)]$

15. (Calculator Permitted) Use your calculator's fnInt(function to evaluate the following integrals. 3 decimials.

    a) $\int_0^5 \frac{x}{x^2+4}dx$ b) $3 + 2\int_0^{\pi/3} \tan x dx$

(G) APPLICATIONS OF DEFINITE INTEGRALS – MOTION PROBLEMS

- 1. An object starts at the origin and moves along the x-axis with a velocity $v(t) = 10t - t^2$ for $0 \leq t \leq 10$
- (a) What is the position of the object at any time, t ?
- (b) What is the position of the object at the moment when it reaches its maximum velocity?
- (c) How far has the object traveled before it starts to decelerate?

(G) APPLICATIONS OF DEFINITE INTEGRALS – MOTION PROBLEMS

- 1. For the velocity functions given below for a particle moving along a line, determine the distance traveled and displacement of the particle:
- (a) $v(t) = 3t - 5, 0 \le t \le 3$
- (b) $v(t) = t^2 - 2t - 8, 1 \le t \le 6$
- 2. The acceleration function and the initial velocity are given for a particle moving along a line. Determine (a) the velocity at time t and (b) the distance traveled during the given time interval:
- (a) $a(t) = t + 4,\ v(0) = 5,\ 0 \le t \le 10$
- (b) $a(t) = 2t + 3, v(0) = -4,\ 0 \le t \le 3$

(G) APPLICATIONS OF DEFINITE INTEGRALS – MOTION PROBLEMS

- Two cars, who are beside each other, leave an intersection at the same instant. They travel along the same road. Car A travels with a velocity of $v(t) = t^2 - t - 6$ m/s while Car B travels with a velocity of $v(t) = 0.5t + 2$ m/s.
- (a) What are the initial velocities of the cars?
- (b) How far has each car gone after 4 seconds have elapsed?
- (c) When are the two cars beside each other again (i.e. when does the trailing car catch up to the leading car?)

(H) AREA BETWEEN CURVES

- Find the area of the regions bounded:
- (a) below $f(x) = x + 2$ and above $g(x) = x^2$
- (b) by $f(x) = 4x$ and $g(x) = x^3$ from x = -2 to x = 2
- (c) Evaluate $\int_{-1}^{3}\left(x^2 - 2x\right)dx$ and interpret the result in

  terms of areas. Then find the area between the graph of $f(x) = x^2 - 2x$ and the x-axis from x = -1 to x = 3

(H) AREA BETWEEN CURVES

- Sketch the curve of $f(x) = x^3 - x^4$ between x = 0 and x = 1.
- (a) Draw a vertical line at $x = k$ such that the region between the curve and axis is divided into 2 regions of equal area. Determine the value of k.
- (b) Draw a horizontal line at $y = h$ such that the region between the curve and axis is divided into 2 regions of equal area. Estimate the value of h. Justify your estimation.

Example #12 - Application

From past records a management services determined that the rate of increase in maintenance cost for an apartment building (in dollars per year) is given by $M'(x) = 90x^2 + 5{,}000$ where M is the total accumulated cost of maintenance for x years.

Application

From past records a management services determined that the rate of increase in maintenance cost for an apartment building (in dollars per year) is given by $M'(x) = 90x^2 + 5{,}000$ where M is the total accumulated cost of maintenance for x years.

Write a definite integral that will give the total maintenance cost through the seventh year. Evaluate the integral.

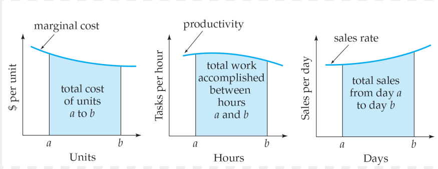
$$\int_0^7 90x^2 + 5{,}000\,dx \qquad 30x^3 + 5{,}000x \Big|_0^7$$

$= 10{,}290 + 35{,}000 - 0 - 0$

$= \$45{,}290$

Total Cost of a Succession of Units

The following diagrams illustrate this idea. In each case, the *curve* represents a *rate*, and the *area under the curve*, given by the definite integral, gives the *total accumulation* at that rate.



FINDING TOTAL PRODUCTIVITY FROM A RATE

A technician can test computer chips at the rate of $-3x^2 + 18x + 15$ chips per hour (for $0 \le x \le 6$), where x is the number of hours after 9:00 a.m. How many chips can be tested between 10:00 a.m. and 1:00 p.m.?

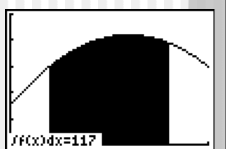
Solution - N (t) = $-3t^2 + 18t + 15$

The total work accomplished is the integral of this rate from $t = 1$ (10 a.m.) to $t = 4$ (1 p.m.):

Use your calculator

$$\int_1^4 \left(-3x^2 + 18x + 15\right) dx$$

$$= \left(-x^3 + 9x^2 + 15x\right)\Big|_{x=1}^{4}$$



= (- 64 + 144 + 60) – (-1 + 9 + 15) = 117

That is, between 10 a.m. and 1 p.m., 117 chips can be tested.

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