LESSON 43 –APPLICATIONS OF DERIVATIVES – Motion, Related Rates and Optimization – DAY 3

Math HL1 - Santowski

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3/7/15 IBHL1 - Calculus - Santowski

LESSON OBJECTIVES

- Apply derivatives to work with rates of change in various contexts: (a) Kinematics, (b) Related Rates (c) Optimization
- Explain what the notation d/dt means
- Given a situation in which several quantities vary, predict the rate at which one of the quantities is changing when you know the other related rates.
- Solve optimization problems using a variety of calculus and non-calculus based strategies

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14. RELATED RATES AND ANGLES

- Example 4
- You walk along a straight path at a speed of 4 m/s. A search light is located on the ground, a perpendicular distance of 20 m from your path. The light stays focused on you. At what rate does the search light rotate when you are 15 meters from the point on the path closest to the search light?

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14. RELATED RATES AND ANGLES

- You walk along a straight path at a speed of 4 m/s. A search light is located on the ground, a perpendicular distance of 20 m from your path. The light stays focused on you. At what rate does the search light rotate when you are 15 meters from the point on the path closest to the search light?
- So we need a relationship between the angle, the 20 meters and your distance along the path ➔ use the primary trig ratios to set this up ➔ the angle is that between the perpendicular (measuring 20 meters) and the path of the opposite side ➔ opposite and adjacent are related by the tangent ratio
- So $\tan(\theta) = x/20$ or $x = 20\tan(\theta)$
- Differentiating ➔ $d/dt\,(x) = d/dt\,(20\tan(\theta))$
- Thus $dx/dt = 20 \times \sec^2(\theta) \times d\theta/dt = 4$ m/s
- Then $d\theta/dt = 4 \div 20\sec^2(\theta)$

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14. RELATED RATES AND ANGLES

- Then $d\theta/dt = 4 \div 20\sec^2(\theta)$
- To find $\sec^2(\theta)$, the measures in our triangle at the instant in question are the 20m as the perpendicular distance, 15m as the distance from the perpendicular, and then the hypotenuse as 25m ➔ so $\sec^2(\theta)$, $= (25/15)^2 = 25/16$
- Then $d\theta/dt = 4 \div (20 \times 25 \div 16) = 0.128$ rad/sec
- So the search light is rotating at 0.128 rad/sec

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15. OPTIMIZING VOLUME

- ex 5. If 2700 cm^2 of material is available to make an open topped box with a square base, find the largest possible volume of the box.
- So once again, decide on what needs to be done
- Outline the steps of your strategy

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15. OPTIMIZING VOLUME

- ex 5. If 2700 cm² of material is available to make an open topped box with a square base, find the largest possible volume of the box.
- Start with what you want to “optimize” ➔ in this case the volume ➔ so we need a volume formula ➔ V = L × W × H
- We know that the base is a square (so L = W) ➔ so we can modify our formula as V = L × L × H = L² × H
- But we are also given info. about the surface area (the material available to us), so we will work with a second relationship
- The surface area of a square based, on top box would be SA = L×L + 4L×H
- Thus 2700 cm² = L² + LH then H = (2700 – L²)/L
- So now we can finalize our formula, before differentiating as:
- V(L) = L² × (2700 – L²)/L = L × (2700 – L²) = 2700L – L³
- But now let’s consider the domain of our function ➔ (0,√2700)

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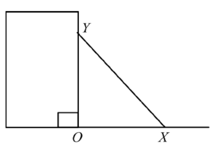
EXAMPLE 16

1982 AB4



A ladder 15 feet long is leaning against a building so that end X is on level ground and end Y is on the wall as shown in the figure. X is moved away from the building at the constant rate of $\frac{1}{2}$ foot per second.

(a) Find the rate in feet per second at which the length OY is changing when X is 9 feet from the building.

(b) Find the rate of change in square feet per second of the area of triangle XOY when X is 9 feet from the building.

EXAMPLE 16

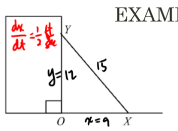
1982 AB 4

$\frac{dy}{dt} = ??$ when $x=9$

$x^2+y^2=15^2$

$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = 0$

$9(1/2) + 12\, dy/dt = 0$



A ladder 15 feet long is leaning against a building so that end X is on level ground and end Y is on the wall as shown in the figure. X is moved away from the building at the constant rate of $\frac{1}{2}$ foot per second.

(a) Find the rate in feet per second at which the length OY is changing when X is 9 feet from the building. $\frac{dy}{dt} = -\frac{9}{24} = \frac{-3}{8}$ ft/sec.

(b) Find the rate of change in square feet per second of the area of triangle XOY when X is 9 feet from the building.

$A = \frac{1}{2}bh = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}x\frac{dy}{dt} + y\frac{1}{2}\frac{dx}{dt}$

$= \frac{1}{2}(9)(-\frac{3}{8}) + (12)(\frac{1}{2})(\frac{1}{2}) = -\frac{27}{16} + \frac{\cancel{12}^{3}}{\cancel{4}} = \frac{48}{16} - \frac{27}{16} = \frac{21}{16}$ ft²/sec

18. OPTIMIZING WITH THREE DIMENSIONAL GEOMETRIC SHAPES

- A can is made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal needed to manufacture the can.
- So once again, decide on what needs to be done
- Outline the steps of your strategy

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18. OPTIMIZING WITH THREE DIMENSIONAL GEOMETRIC SHAPES

- A can is made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal needed to manufacture the can.
- we wish to minimize the surface area, so the surface area formula for cylinders is the place to start
- $SA(r,h) = 2\pi r^2 + 2\pi rh = 2\pi r(r + h)$
- Now again, we have 2 variables, so we need to express one in terms of the other
- The volume of a cylinder is $V = \pi r^2 h = 1000\ cm^3 = 1L$
- So $1000/\pi r^2 = h$
- Therefore $SA(r) = 2\pi r(r + 1000/\pi r^2) = 2\pi r^2 + 2000r^{-1}$
- And the domain would be? Well SA > 0, so $2\pi r^2 + 2000r^{-1} > 0$

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18. OPTIMIZING WITH THREE DIMENSIONAL GEOMETRIC SHAPES

- A cylindrical can is made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal needed to manufacture the can.
- Now differentiate to find the critical points on the SA function
- $SA`(r) = 4\pi r - 2000r^{-2} = 0 \rightarrow r^3 = 500/\pi \rightarrow r = 5.42\ cm \rightarrow h = 10.8\ cm$
- Now let's run through the first derivative test to verify a minimum SA
- $SA`(5) = 4\pi(5) - 2000(5)^{-2} = -17.2$
- $SA`(6) = 4\pi(6) - 2000(6)^{-2} = 19.8$
- So the first derivative changes from –ve to +ve, meaning that the value r = 5.42 cm does represent a minimum

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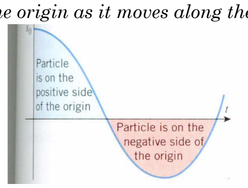
EX 19 OPTIMIZING WITH THREE DIMENSIONAL GEOMETRIC SHAPES

- A cylindrical can is made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal needed to manufacture the can but the following conditions are included: the cost of the bottom is twice as much as the cost of the "sides" and the cost of the top is twice the cost of the bottom.

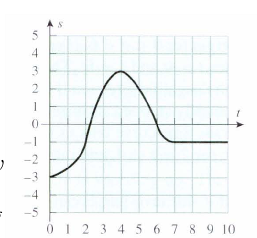
RECTILINEAR MOTION

- *The figure below is a typical position vs. time curve for a particle in rectilinear motion. We can tell from the graph that the coordinate of the particle at $t = 0$ is s_0, and we can tell from the sign of s when the particle is on the negative or the positive side of the origin as it moves along the coordinate line.*



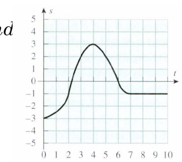
EXAMPLE 20

- *The figure below shows the position vs. time curve for a particle moving along an s-axis. In words, describe how the position of the particle is changing with time.*



EXAMPLE 20 - soln

- *The figure below shows the position vs. time curve for a particle moving along an s-axis. In words, describe how the position of the particle is changing with time.*
- *At t = 0, s(t) = -3. It moves in a positive direction until t = 4 and s(t) = 3. Then, it turns around and travels in the negative direction until t = 7 and s(t) = -1. The particle is stopped after that.*



VELOCITY AND SPEED

- *The rate of change of your position is based on your velocity. The rate of change is the first derivative. This leads us to:*

$$v(t) = s^{/}(t) = \frac{ds}{dt}$$

- *Remember, velocity has direction attached to it. If velocity is positive, the particle is moving to the right or up. If velocity is negative, the particle is moving to the left or down.*
- *Speed is just how fast you are going regardless of direction. This leads us to:* $|v(t)| = |s^{/}(t)| = speed$

EXAMPLE 21

- *Let s(t) = $t^3 - 6t^2$ be the position function of a particle moving along the s-axis, where s is in meters and t is in seconds. Find the velocity and speed functions, and show the graphs of position, velocity, and speed versus time.*

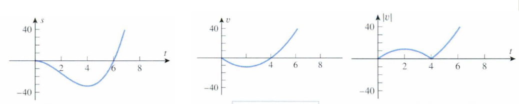
EXAMPLE 21 - soln

- *Let $s(t) = t^3 - 6t^2$ be the position function of a particle moving along the s-axis, where s is in meters and t is in seconds. Find the velocity and speed functions, and show the graphs of position, velocity, and speed versus time.*

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t \quad speed = |v(t)| = |3t^2 - 12t|$$

EXAMPLE 21 - soln

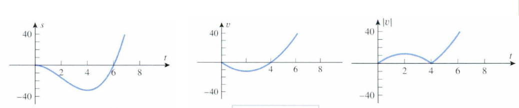
- *The graphs below provide a wealth of visual information about the motion of the particle.*
- *Describe what information the three graphs present to us about the motion of the particle*



Position versus time | Velocity versus time | Speed versus time

EXAMPLE 21 - soln

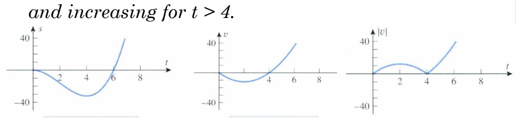
- *The graphs below provide a wealth of visual information about the motion of the particle. For example, the position vs. time curve tells us that the particle is on the negative side of the origin for $0 < t < 6$, is on the positive side of the origin for $t > 6$ and is at the origin at times $t = 0$ and $t = 6$.*



Position versus time | Velocity versus time | Speed versus time

EXAMPLE 21 - soln

- *The velocity vs. time curve tells us that the particle is moving in the negative direction if $0 < t < 4$, and is moving in the positive direction if $t > 4$ and is stopped at times $t = 0$ and $t = 4$ (the velocity is zero at these times). The speed vs. time curve tells us that the speed of the particle is increasing for $0 < t < 2$, decreasing for $2 < t < 4$ and increasing for $t > 4$.*



Position versus time   Velocity versus time   Speed versus time

ACCELERATION

- *The rate at which the instantaneous velocity of a particle changes with time is called* <u>*instantaneous acceleration.*</u> *We define this as:*

$$a(t) = v'(t) = s''(t) = \frac{dv}{dt}$$

- *We now know that the first derivative of position is velocity and the second derivative of position is acceleration.*

EXAMPLE 22

- *Let $s(t) = t^3 - 6t^2$ be the position function of a particle moving along an s-axis where s is in meters and t is in seconds. Find the acceleration function a(t) and show that graph of acceleration vs. time.*

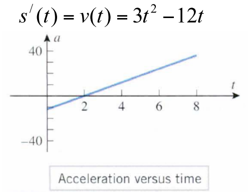
EXAMPLE 22

- *Let s(t) = t³ – 6t² be the position function of a particle moving along an s-axis where s is in meters and t is in seconds. Find the acceleration function a(t) and show that graph of acceleration vs. time.*

$s'(t) = v(t) = 3t^2 - 12t$

$s''(t) = a(t) = 6t - 12$

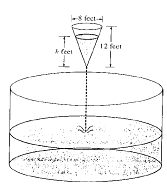
$6(t-2)$



Acceleration versus time

1995 AB5/BC3

EXAMPLE 23



As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h.

(b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.

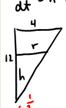
(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure.

Video

1995 AB5/BC3

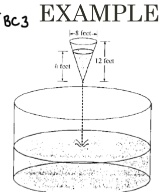
EXAMPLE 23

$\frac{dh}{dt} = h - 12$



$\frac{\cancel{4}^{\,1}}{\cancel{12}_{\,3}} = \frac{r}{h}$

$3r = h$ or $r = \frac{1}{3}h$



$V = \frac{1}{3}\pi r^2 h$
$= \frac{\pi}{3}\left(\frac{1}{3}h\right)^2 h$
$= \frac{\pi}{3}\left(\frac{1}{9}h^2\right)h$
$= \boxed{\frac{\pi}{27}h^3}$

$\frac{dV}{dt} = \frac{3\pi}{27}h^2\frac{dh}{dt}$
$= \cancel{\frac{3\pi}{27}}(\cancel{3})^{\cancel{2}}(-9)$
$= \boxed{-9\pi \text{ ft}^3/\text{min}}$

$V = Bh$
$V = 400\pi y$
$\frac{dV}{dt} = 400\pi\frac{dy}{dt}$
$9\pi = 400\pi \boxed{\frac{dy}{dt}}$
$\frac{dy}{dt} = \boxed{\frac{9}{400} \text{ ft}^3/\text{min}}$

As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h.

$V = (\pi/27)h^3$

(b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure. -9π ft³/min

(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure. $9/400$ ft³/min

24. OPTIMIZING WITH THREE DIMENSIONAL GEOMETRIC SHAPES

- Ex 9. Naomi is designing a cylindrical can which will hold 280 mL of juice. The metal for the sides costs \$0.75/m^2 while the metal for the top and bottom costs \$1.40/m^2. Find the dimensions that will minimize the cost of the materials.

- Ex 10. A tank with a rectangular base and rectangular sides is open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs \$10/sq. m. for the base and \$5/sq. m. for the sides, what is the cost of the least expensive tank, and what are its dimensions?

EXAMPLE 25

1990 AB4

The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second. (Note: The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.)

(a) At the time when the radius of the sphere is 10 centimeters, what is the rate of increase of its volume?

(b) At the time when the volume of the sphere is 36π cubic centimeters, what is the rate of increase of the area of a cross section through the center of the sphere?

(c) At the time when the volume and the radius of the sphere are increasing at the same numerical rate, what is the radius?

EXAMPLE 25

Video

1990 AB 4

The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second. (Note: The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.) $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(100)\left(\frac{4}{100}\right) = 16\pi$ cm^3/sec.

(a) At the time when the radius of the sphere is 10 centimeters, what is the rate of increase of its volume? 16π cm^3/sec.

(b) At the time when the volume of the sphere is 36π cubic centimeters, what is the rate of increase of the area of a cross section through the center of the sphere? $.24\pi$ cm^2/sec.

(c) At the time when the volume and the radius of the sphere are increasing at the same numerical rate, what is the radius? $\frac{1}{2\sqrt{\pi}}$

$V = \frac{4}{3}\cancel{\pi} r^3 = 36\cancel{\pi}$

$r^3 = 27$

$r = 3$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

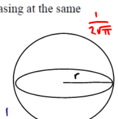
$\frac{dA}{dt} = 2\pi(3)\left(\frac{4}{100}\right) = \frac{24\pi}{100} = .24\pi \frac{cm^2}{sec.}$

when $\frac{dV}{dt} = \frac{dr}{dt}$, $r = ??$

$4\pi r^2 \frac{dr}{dt} = \frac{dr}{dt}$

$4\pi r^2 = 1$

$r^2 = \frac{1}{4\pi}$ $\quad r = \frac{1}{\sqrt{4\pi}} = \frac{1}{2\sqrt{\pi}}$



SPEEDING UP AND SLOWING DOWN

- *We will say that a particle in rectilinear motion is speeding up when its speed is increasing and slowing down when its speed is decreasing. In everyday language an object that is speeding up is said to be "accelerating" and an object that is slowing down is said to be "decelerating."*
- *Whether a particle is speeding up or slowing down is determined by both the velocity and acceleration.*

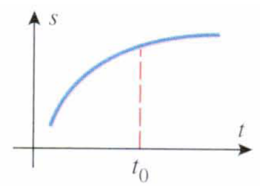
THE SIGN OF ACCELERATION

- *A particle in rectilinear motion is speeding up when its velocity and acceleration have the same sign and slowing down when they have opposite signs.*

ANALYZING THE POSITION VS. TIME CURVE

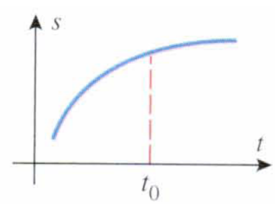
- *If $s(t) > 0$, the particle is on the positive side of the s-axis and if $s(t) < 0$, the particle is on the negative side of the s-axis.*
- *The slope of the curve at any time is equal to the instantaneous velocity at that time.*
- *Where the curve has positive slope, the velocity is positive and the particle is moving in the positive direction.*
- *Where the curve has negative slope, the velocity is negative and the particle is moving in the negative direction.*
- *Where the curve has slope zero, the velocity is zero and the particle is momentarily stopped.*

POSITION VS. TIME CURVE



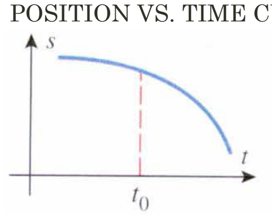
- From this position vs. time curve determine if at t_0:
  - (a) The particle is on the positive or negative side of the origin.
  - (b) The direction the particle is moving.
  - (c) The particle is speeding up or slowing down.

POSITION VS. TIME CURVE



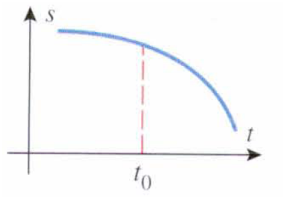
- The particle is on the positive side of origin
- 1st derivative is positive, so the particle is moving in the positive direction
- 2nd derivative is negative (concave down), so the particle is slowing down.

POSITION VS. TIME CURVE



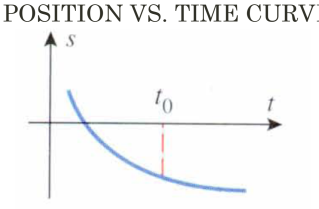
- From this position vs. time curve determine if at t_0:
  - (a) The particle is on the positive or negative side of the origin.
  - (b) The direction the particle is moving.
  - (c) The particle is speeding up or slowing down.

POSITION VS. TIME CURVE



- The particle is on the positive side of origin
- 1^{st} derivative is negative, so the particle is moving in the negative direction
- 2^{nd} derivative is negative, so the particle is speeding up.

POSITION VS. TIME CURVE



- From this position vs. time curve determine if at t_0:
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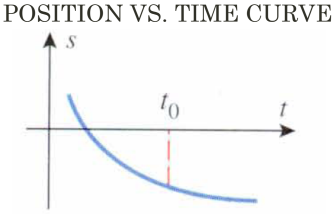
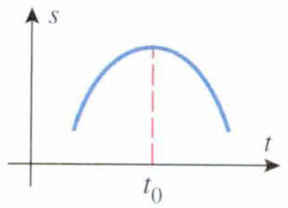
  - (b) The direction the particle is moving.
  - (c) The particle is speeding up or slowing down.

POSITION VS. TIME CURVE



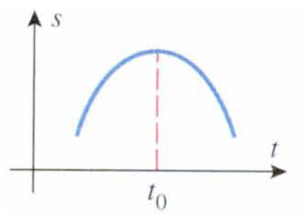
- The particle is on the negative side of origin
- 1^{st} derivative is negative, so the particle is moving in the negative direction
- 2^{nd} derivative is positive, so the particle is slowing down.

POSITION VS. TIME CURVE



- *From this position vs. time curve determine if at t_0:*
  - *(a) The particle is on the positive or negative side of the origin.*
  - *(b) The direction the particle is moving.*
  - *(c) The particle is speeding up or slowing down.*

POSITION VS. TIME CURVE



- *The particle is on the positive side of origin*
- *The particle is momentarily stopped.*
- *The velocity is decreasing.*

26. KINEMATICS

- Ex 2 – The position of a particle moving on a line is modeled by $\boldsymbol{s(t)} = 2\boldsymbol{t}^3 - 21\boldsymbol{t}^2 + 60\boldsymbol{t}$, $\boldsymbol{t} > 0$, where $\boldsymbol{t}$ is measured in seconds and $\boldsymbol{s}$ in meters.
- (a) What is the velocity after 3 s and after 6 s?
- (b) When is the particle at rest?
- (c) When is the particle moving in the positive direction?
- (d) Find the total distance traveled by the particle during the first 6 seconds

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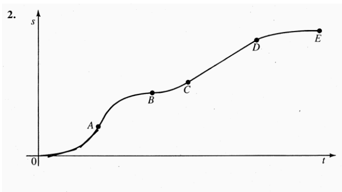
27. KINEMATICS



The graph of a position function is shown.

(a) For the part of the graph from O to A, use slopes of tangents to decide whether the velocity is increasing or decreasing. Is the acceleration positive or negative?

(b) State whether the acceleration is positive, zero, or negative
(i) from A to B (ii) from B to C
(iii) from C to D (iv) from D to E

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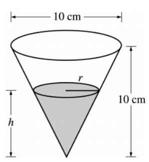
2002 AP® CALCULUS AB FREE-RESPONSE QUESTIONS

EXAMPLE 28



5. A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

(Note: The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)

(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

EXAMPLE 28

Video

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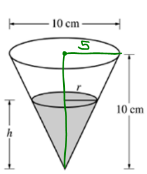
$\frac{dV}{dt} = k\pi r^2$

$-\frac{75\pi}{40} = k\frac{25\pi}{4}$

$k = \frac{-75\pi}{40}\left(\frac{4}{25\pi}\right)$

$= -\frac{3}{10}$



$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\left(\frac{1}{2}h\right)^2 h = \frac{\pi}{12}h^3$

$\frac{dV}{dt} = \frac{3\pi}{12}h^2\frac{dh}{dt}$

$= \frac{\pi}{4}(25)\left(-\frac{3}{10}\right) = \boxed{\frac{-75\pi}{40}\frac{cm^3}{hr}}$

A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr. $\frac{dh}{dt} = -\frac{3}{10}$ cm/hr. $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\left(\frac{5}{2}\right)^2 5 = \boxed{\frac{125\pi}{12} cm^3}$

(Note: The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.) $\frac{5}{10} = \frac{r}{h}$ $10r = 5h$ $r = \frac{1}{2}h$

(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure. $\frac{125}{12}$ cm^3

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure. $\frac{dV}{dt} = ??$ when $h=5$ $-75\pi/40$ cm^3/hr.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

EXAMPLE 29

- ex 13. A north-south highway intersects an east-west highway at point P. A vehicle passes point P at 1:00 pm traveling east at a constant speed of 60 km/h. At the same instant, another vehicle is 5 km north of P, traveling south at 80 km/h. Find the time when the two vehicles are closest to each other and the distance between them at that time.
- Show how to come up with a distance equation of $d^2 = (5 - 80t)^2 + (60t)^2$

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EXAMPLE 30

1. The position of a skateboarder at any time t (in seconds) is given by the function $s(t) = t^3 - 8t^2 + 8t$ measured in feet.
   a) What are the velocity and acceleration functions in terms of t?
   b) When is the skateboarder at rest?
   c) What is the position(s) of the skateboarder when at rest?
   d) What are the position, velocity, and acceleration of the skateboarder at three seconds and at 5 seconds?
   e) Sketch a motion schematics for the skateboarder. Make sure to label position and velocity at each critical time.
   f) What was the total distance traveled in the first five seconds?
   g) Find the displacement in the first 5 seconds.
   h) When is the skateboarder moving to the right and left? Use interval notation for your answers.

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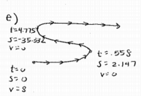
EXAMPLE 30 - SOLN

a) $v(t) = 3t^2 - 16t + 8$
$a(t) = 6t - 16$

b) $0 = 3t^2 - 16t + 8$
$t = .558$ sec
$t = 4.775$ sec

c) $s(.558) = 2.147$ ft.
$s(4.775) = -35.332$ ft.

d) $s(3) = -21$ ft $s(5) = -35$ ft
$v(3) = -13$ ft/sec $v(5) = 3$ ft/sec
$a(3) = 2$ ft/sec^2 $a(5) = 14$ ft/sec^2

e)


f) s=0 s=2.147 s=-35.332 s=-35
0 .558 4.775 5
$|2.147| + |-37.479| + |.332| =$
39.958 ft.

g) $s(5) - s(0)$
$-35 - 0 = -35$ ft.
35 ft. to the left

h) left: $(.558, 4.775)$
right: $[0, .558) \cup (4.775, \infty)$

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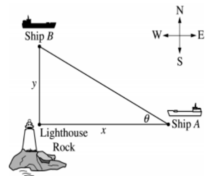
EXAMPLE 31



6. Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

(a) Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km.

(b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

(c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km.

EXAMPLE 31

Video

2002 AB 6 Form B (No Calculator)

$\tan\theta = y/x$

$\sec^2\theta \frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$

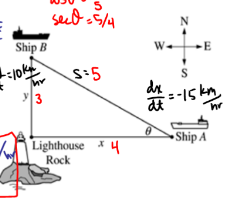
$\cos\theta = \frac{4}{5}$

$\sec\theta = 5/4$

$\frac{25}{16}\frac{d\theta}{dt} = \frac{4(10)-3(-15)}{16}$ $\quad \frac{dy}{dt} = 10\frac{km}{hr}$

$\frac{25}{16}\boxed{\frac{d\theta}{dt}} = \frac{85}{16}$

$\frac{d\theta}{dt} = \frac{85}{16}\cdot\frac{16}{25} = \frac{85}{25} = \boxed{\frac{17}{5}\ \text{rad/hr}}$



$\frac{dx}{dt} = -15\frac{km}{hr}$

$x^2 + y^2 = s^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$

$x\frac{dx}{dt} + y\frac{dy}{dt} = s\boxed{\frac{ds}{dt}}$

$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{s}$

$= \frac{4(-15)+3(10)}{5}$

$= \frac{-60+30}{5} = \boxed{-6\frac{km}{hr}}$

Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

(a) Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km. 5km

(b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km. $-6\frac{km}{hr}$

(c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km. 17/5 rad/hr

EXAMPLE 32

- ex 14. I want to run a power line to a cottage being built on an island that is 400 m from the shore of a lake. The main power line ends 3 km away from the point on the shore that is closest to the island. The cost of laying the power line underwater is twice the cost of laying the power line on the land. How should I place the power line so that I minimize the overall cost?
- Show how to develop the equation $C = 2L_1 + L_2 = 2(\text{sqrt}(x^2 + 160{,}000)) + (3000 - x)$

EXAMPLE 33

2. The position of a particle at any time t (in seconds) is given by the function $s(t) = 2t^3 - 27t + 15$ measured in feet.
    a) What are the velocity and acceleration equation in terms of t?
    b) When is the particle at rest?
    c) What is the position(s) of the particle when it is at rest?
    d) What are the initial position, velocity, and acceleration of the particle?
    e) Sketch a motion schematics for the skateboarder. Make sure to label position and velocity at each critical time.
    f) What is the total distance traveled from one second to six seconds?
    g) What is the displacement for the same time frame?
    h) When is the particle moving to the left and right? Use interval notation for your answers.

EXAMPLE 33 – SOLN

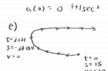
a) $v(t) = 6t^2 - 27$
$a(t) = 12t$

b) $0 = 6t^2 - 27$
$27 = 6t^2$
$\sqrt{t^2} = \sqrt{4.5}$
$t = 2.121$ sec

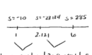
c) $s(2.121) = -23.184$ ft.

d) $s(0) = 15$ ft
$v(0) = -27$ ft/sec
$a(0) = 0$ ft/sec^2

e)



f)



$|-13.184| + |308.184| =$
321.368 ft.

g) $s(6) - s(1)$
$285 - (-10)$
295 ft.

h) left: $[0, 2.121)$
right: $(2.121, \infty)$

EXAMPLE 34

- A particle's displacement, x meters from an origin O is defined by following equation at time t seconds ➔ $x(t) = \sin(4t) - \cos(4t)$
    - Show that the particle's acceleration is proportional to its displacement.
    - What is the particle's maximum speed?
    - Describe the particle's motion.

EXAMPLE 35

- An offshore well is located in the ocean at a point W which is six miles from the closest shore point A on a straight shoreline. The oil is to be piped to a shore point B that is eight miles from A by piping it on a straight line under water from W to some shore point P between A and B and then on to B via a pipe along the shoreline. If the cost of laying pipe is $100,000 per mile under water and $75,000 per mile over land, how far from A should the point P be located to minimize the cost of laying the pipe? What will the cost be?

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EXAMPLE 36

3. A particle is moving with its position defined by $s(t) = t^3 - 6t^2 + 9t + 5$ where t is in seconds and s is in feet.
   a) What are the particle's velocity and acceleration functions?
   b) Find the displacement and the total distance traveled by the particle in the first four seconds.
   c) What is the velocity of the particle when its position is 8 feet?
   d) Sketch a motion schematic labeling position, velocity, and acceleration at the beginning, end, and at each change.

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EXAMPE 36 – SOLN

a) $v(t) = 3t^2 - 12t + 9$
$a(t) = 6t - 12$

b) $0 = 3t^2 - 12t + 9$
$0 = 3(t^2 - 4t + 3)$
$0 = 3(t-3)(t-1)$
$t = 3 \quad t = 1$

s=5 s=9 s=5 s=9
0 1 3 4

$|9-5| + |5-9| + |9-5|$
$4 + 4 + 4 = 12$ ft
↳ total distince

$s(4) - s(0)$
$9 - 5 = 4$ ft
↳ displacement

c) $8 = t^3 - 6t^2 + 9t + 5$
$0 = t^3 - 6t^2 + 9t - 3$
$t = .468$ sec
$t = 1.653$ sec
$t = 3.879$ sec

$v(.468) = 4.041$ ft/sec
$v(1.653) = -2.639$ ft/sec
$v(3.879) = 7.592$ ft/sec

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EXAMPLE 37

- ex 14. Which points on the graph of $y = 9 - x^2$ are closest to the point (0,6)?
- Show how to develop the equation $d^2 = (x - 0)^2 + (9 - x^2 - 6)^2$