# LESSON 43 –APPLICATIONS OF DERIVATIVES – Motion, Related Rates and Optimization

**Math HL1 - Santowski**

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## LESSON OBJECTIVES

- Apply derivatives to work with rates of change in various contexts: (a) Kinematics, (b) Related Rates (c) Optimization
- Explain what the notation d/dt means
- Given a situation in which several quantities vary, predict the rate at which one of the quantities is changing when you know the other related rates.
- Solve optimization problems using a variety of calculus and non-calculus based strategies



## 1. KINEMATICS (CI)

- Ex 1 – If a stone is dropped from a cliff that is 122.5 meters high, then its height, in meters, after $t$ seconds is $h(t) = 122.5 - 4.9t^2$.
- (a) Find the velocity of the stone at $t$ = 1.0 sec and at $t$ = 2.0 seconds.
- (b) When will the stone hit the ground?
- (c) With what velocity will the stone hit the ground?

## 2. KINEMATICS (CA)

- Ex 2 – A projectile is launched upwards and experiences resistance such that its height above the ground after a time of ***t*** seconds is modeled by ***h***(***t***) = $220(1 - e^{-0.2t}) - 20t$
- (a) Find the initial velocity of the projectile.
- (b) What is the maximum height of the projectile?
- (c) With what velocity will the stone hit the ground?

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## 3. RELATED RATES

- 1. Determine dy/dt for $y(t) = t^2$
- 2. Determine dy/dt for $y(x) = x^2$ where x(t)
- 3. Give a practical meaning for dy/dx
- 4. Give a practical meaning for dV/dt
- 5. Give a practical meaning for dV/dr
- 6. Determine dV/dr if $V = 1/3\pi r^2h$
- 7. Determine dV/dh if $V = 1/3\pi r^2h$
- 8. Determine dV/dt if $V = 1/3\pi r^2h$

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## REVIEW

- Recall that the meaning of d*y*/d*x* is a change of *y* as we change *x*
- Recall that a rate can also be understood as a change of some quantity with respect to time ➔ As a derivative, we would write this as d*X* / d*t*
- So therefore d*V* /d*t* would mean ➔ the rate of change of the volume as time changes
- Likewise d*A*/ d*r* ➔ the rate of change of the area as we make changes in the radius
- So also consider:
  - d*V* / d*r* ➔
  - d*h* / d*r* ➔
  - d*h* / d*t* ➔
  - d*r* / d*t* ➔

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## 4. Related Rates and Circles



- example 1
- A pebble is dropped into a pond and the ripples form concentric circles. The radius of the outermost circle increases at a constant rate of 10 cm/s. Determine the rate at which the area of the disturbed water is changing when the radius is 50 cm.



## 4. Related Rates and Circles (soln)



- ex 1. A pebble is dropped into a pond and the ripples form concentric circles. The radius of the outermost circle increases at a constant rate of 10 cm/s. Determine the rate at which the area of the disturbed water is changing when the radius is 50 cm.
- So, as the radius changes with time, so does the area ➔ dr/dt is related to dA/dt ➔ how?
- Recall the area formula ➔ $A = \pi r^2$ ➔ $A(t) = \pi(r(t))^2$
- Then use implicit differentiation as we now differentiate with respect to time ➔ $d/dt\ (A(t)) = d/dt\ (\pi(r(t))^2) = 2\pi r \times dr/dt$
- So now we know the relationship between dA/dt and dr/dt
- Then if we knew dr/dt, we could find dA/dt
- It is given that dr/dt is 10 cm/sec, then $dA/dt = 2 \times \pi \times (50\text{cm}) \times 10$ cm/sec
- $dA/dt = 1000\pi\ \text{cm}^2/\text{sec}$ or 3141 square cm per second



## 5. Optimization



- 1. Find the maximum point of $M = x(30-2x)^2$ on [0,30]
- 2. If $S(r,h) = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h = 1000$, find an expression for S(r)
- 3. An 8 cm piece of wire is to be cut into 2 pieces, one to be formed into a circle and one to be formed into a square. Find an expression for the total area enclosed by the two figures

## 6. Optimizing Area - Example

- **ex 1. A farmer wishes to enclose a rectangular lot with area being 600 $m^2$. What is the least amount of fencing that will be needed?**
- First, before we start with any "math", you will provide a list of "what needs to be done and why"

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## 6. Optimizing Area – Example SOLN

- **ex 1. A farmer wishes to enclose a rectangular lot each being 600 $m^2$. What is the least amount of fencing that will be needed?**
- Start with key relationships ➔ $p = 2(L + W)$ and $A = L \times W$
- The reason for looking for a second relationships is that my first relationship (perimeter) has two variables, so I would want to make a substitution giving me a relationship with only one variable
- Since I wish to optimize the perimeter ➔ $p = 2(L + W)$
- Now I use the relationship $A = 600 = L \times W$ ➔ so $W = 600/L$
- Then $p(L) = 2L + 2(600/L) = 2L + 1200L^{-1}$

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## 6. Optimizing Area – Example SOLN

- **ex 1. A farmer wishes to enclose a rectangular lot each being 600 $m^2$. What is the least amount of fencing that will be needed?**
- Given that $p(L) = 2L + 2(600/L) = 2L + 1200L^{-1}$
- Since we want to "optimize" the fencing (use the least amount), we want to find the minimum point on the graph of $y = p(L)$
- So we will differentiate and set the derivative to 0
- $d/dL\,(p(L)) = d/dL\,(2L + 1200L^{-1}) = 2 - 1200L^{-2}$
- So $p`(L) = 0 = 2 - 1200L^{-2}$ ➔ thus $L^2 = 600$ or $L = \sqrt{600} = 24.5$ m

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## 6. Optimizing Area – Example SOLN

- Now, we need to verify that the critical point that we generated (when L = 24.5m) is actually a minimum point.
- We can use the first derivative test and we can substitute L = 20m and L = 25m into the derivative equation
- $p`(20) = 2 - 1200(20)^{-2} = 2 - 3 = -1$
- $p`(25) = 2 - 1200(25)^{-2} = 2 - 1.92 = +0.08$
- So our derivative changes from –ve to +ve as L increases, so our function must have a minimum point at 24.5 m

- Alternatively, we can use the second derivative test
- $p``(L) = -1200 \times -2L^{-3} = 2400/L^3$
- Then $p``(24.5) = 2400/(24.5^3) = +0.163$ ➔ which means that the second derivative is positive at L = 24.5m meaning our function is concave up which means that our critical value at L = 24.5 must correspond to a minimum



Strategy for solving max/min problems:

1. Understand the problem.
2. Develop a mathematical model of the problem
   a) sketch a diagram
   b) introduce variables to represent unknown quantities
   c) write an equation for the quantity to be maximized or minimized (optimized)
   d) write any additional equations that allow the dependent variables to be related; use it/them to reduce the number of independent unknowns in the optimizing function down to one variable; these relating equations often come from the problem statement itself, or the geometry of the diagram.
3. Determine the domain of the function.
4. Identify the critical points and the endpoints.
5. Use the SDT or the FDT to identify the candidate(s) as the x – coordinate of a maximum or minimum of the function to be optimized
6. Solve the mathematical model.
7. Interpret the solution.



## 7. Related Rates and 3D Volumes

- Example 2.

- A water tank has a shape of an inverted circular cone with a base radius of 2 m and a height of 4 m. If water is being pumped in a rate of 2 $m^3$/min, find the rate at which the water level is rising when the water is 3 meters deep.

## 7. Related Rates and 3D Volumes



- Ex 2. A water tank has a shape of an inverted circular cone with a base radius of 2 m and a height of 4 m. If water is being pumped in a rate of 2 $m^3$/min, find the rate at which the water level is rising when the water is 3 meters deep.
- So in the context of this conical container filling, we see that the rate of change of the volume is related to 2 different rates ➔ the rate of change of the height and the rate of change of the radius
- Likewise, the rate of change of the height is related to the rate of change of the radius and the rate of change of time.
- So as we drain the conical container, several things change ➔ V,r,h,t and how they change is related

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## 7. Related Rates and 3D Volumes



- Having the formula V = $1/3\pi r^2 h$ (or recall that $V(t) = 1/3\pi(r(t))^2h(t)$) and the differentiation $dV/dt = d/dt(1/3\pi r^2 h)$, we can now take the derivative
- $dV/dt = 1/3\pi \times d/dt\,[(r^2 \times h)]$
- $dV/dt = 1/3\pi \times [\,(2r \times dr/dt \times h) + (dh/dt \times r^2)]$
- So as we suspected initially, the rate at which the volume in a cone changes is related to the rate at which the radius changes and the rate at which the height changes.
- If we know these 2 rates (dr/dt and dh/dt) we can solve the problem
- But if we do not know the 2 rates, we need some other relationship to help us out.
- In the case of a cone ➔ we usually know the relationship between the radius and the height and can express one in terms of the other

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## 7. Related Rates and 3D Volumes



- In a right angled cone (radius is perpendicular to the height) the ratio of height to radius is always constant
- In this case, h/r = 4/2 ➔ so r = ½h
- So our formula $V = 1/3\pi r^2 h$ becomes $V = 1/3\pi(½h)^2 h = 1/12\pi h^3$
- Now we can differentiate again
- $dV/dt = d/dt\,(1/12\pi h^3)$
- $dV/dt = 1/12\pi \times 3h^2 \times dh/dt$
- $2\ m^3/min = 1/12\pi \times 3(3)^2 \times dh/dt$
- $dh/dt = 2 \div (27/12\pi)$
- $dh/dt = 8\pi/9$ m/min

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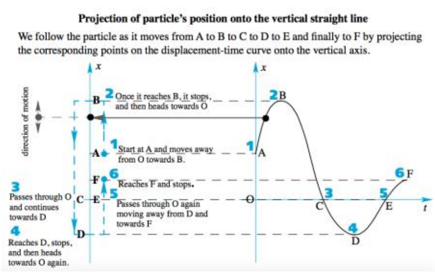
## Rectilinear Motion

- *A particle that can move in either direction along a coordinate line is said to be in rectilinear motion. The line could be the x-axis or the y-axis. We will designate the coordinate line as the s-axis.*
- *The position function of the particle is s(t) and we call the graph of s versus t the position vs. time curve.*
- *The change in the position of the particle is called the displacement of the particle. The displacement describes where it is compared to where it started.*

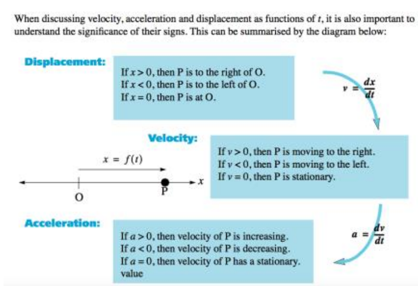
## Rectilinear motion - basics





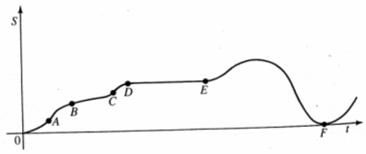



## Rectilinear motion - basics



## 8. KINEMATICS

**1.** The graph shows the position function of a car.




(a) What was the initial velocity of the car?
(b) Was the car going faster at $B$ or at $C$?
(c) Was the car slowing down or speeding up at $A$, $B$, and $C$?
(d) What happened between $D$ and $E$?
(e) What happened at $F$?

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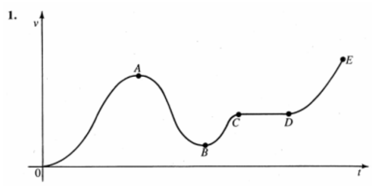
## 9. KINEMATICS

**1.**




The graph of a velocity function is shown. State whether the acceleration is positive, zero, or negative

(a) from $O$ to $A$,
(b) from $A$ to $B$,
(c) from $B$ to $C$,
(d) from $C$ to $D$,
(e) from $D$ to $E$.

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## 10. KINEMATICS

- A particle moves in a straight line with a displacement of s meters t seconds after leaving a fixed point. The displacement function is given by .
- i. Find the velocity of the particle at any time, t.
- ii. Find the initial position and initial velocity of the particle.
- iii. Find when the particle is at rest.
- iv. Find when the particle is moving left and when it is moving right.
- v. Draw a motion diagram for the particle.
- vi. Find the average velocity for the first 3 seconds of motion.
- vii. Find the average acceleration of the particle from t = 1 sec and t = 4 sec.
- viii. Find the instantaneous acceleration at t = 3 seconds. Explain the meaning of your answer.
- ix. Find the speed of the particle at t = 3 sec and determine the particle is speeding up or slowing down at that time.
- x. During 0 < t < 10 seconds, find the intervals when the particle is speeding up or slowing down.

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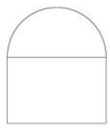
## 11. OPTIMIZING AREA – QUESTIONS



**Example**

A Norman window has the outline of a semicircle on top of a rectangle. Suppose there is $8 + 4\pi$ feet of wood trim available. Discuss why a window designer might want to maximize the area of the window. Find the dimensions of the rectangle and semicircle that will maximize the area of the window.





## 11. OPTIMIZING AREA – QUESTIONS



**Solution**

*We have to maximize* $A = \ell w + (w/2)^2\pi$ *subject to the constraint that* $2\ell + w + \pi w = p$. *Solving for* $\ell$ *in terms of* $w$ *gives*

$$\ell = \tfrac{1}{2}(p - w - \pi w)$$

*So* $A = \frac{1}{2}w(p - w - \pi w) + \frac{1}{4}\pi w^2$. *Differentiating gives*

$$A'(w) = \frac{\pi w}{2} + \frac{1}{2}(-1 - \pi)w + \frac{1}{2}(p - \pi w - w)$$

*which is zero when* $w = \dfrac{p}{2+\pi}$. *If* $p = 8 + 4\pi$, $w = 4$. *It follows that* $\ell = 2$.



## 12. PYTHAGOREAN RELATIONSHIPS



- Example 3
- A ladder 10 meters long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 m from the foot of the wall?

## 12. Pythagorean Relationships



- A ladder 10 meters long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 m from the foot of the wall?
- If we set up a diagram, we create a right triangle, where the ladder represents the hypotenuse and then realize that the quantities that change with time are the distance of the foot of the ladder along the floor (*x*) and the distance from the top of the ladder to the floor (*y*)
- So we let x represent this distance of the foot of the ladder and y represents the distance of the top of the ladder to the floor
- So our mathematical relationship is $x^2 + y^2 = 10^2$



## 12. Pythagorean Relationships



- So we have $x^2 + y^2 = 10^2$ and we simply differentiate wrt time
- Then $d/dt\,(x^2 + y^2) = d/dt\,(10^2)$
- $2x \times dx/dt + 2y \times dy/dt = 0$
- $x \times dx/dt = - y \times dy/dt$
- Then $dy/dt = x \times dx/dt \div - y$
- So $dy/dt = (6\text{ m}) \times (1\text{ m/s}) \div -(8) =$
- $dy/dt = -3/4$ m/sec
- So the top of the ladder is coming down at a rate of 0.75 m/sec when the foot of the ladder is 6 m from the wall



## 13. KINEMATICS

- *Suppose that the position function of a particle moving on a coordinate line is given by* $$s(t) = 2t^3 - 21t^2 + 60t + 3$$
- *Analyze the motion of the particle.*

## 13. KINEMATICS - SOLN

$$s(t) = 2t^3 - 21t^2 + 60t + 3$$

$$v(t) = 6t^2 - 42t + 60 = 6(t-2)(t-5)$$

*right* *left* *right*

|___+___|_____-_____|___+___

*0* *2* *5*

*stop* *stop*

$$a(t) = 12t - 42 = 12(t - 3.5)$$

|_____-_______|_____+_______

*0* *3.5*

## 13. KINEMATICS - SOLN

*right* *left* *right*

|___+___|_____-_____|___+___

*0* *2* *5*

|__-_____|__-_|__+__|__+_____

*0* *2* *3.5* *5*

- *From 0-2, moving to the right and slowing down.*
- *From 2-3.5, left, speeding up.*
- *From 3.5-5, left, slowing down.*
- *From 5 on, right and speeding up.*

## 13. KINEMATICS - SOLN

- *From 0-2, moving to the right and slowing down.*
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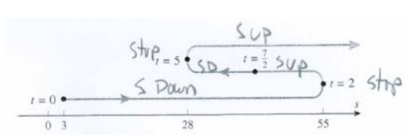

## 14. Related Rates and Angles

- Example 4
- You walk along a straight path at a speed of 4 m/s. A search light is located on the ground, a perpendicular distance of 20 m from your path. The light stays focused on you. At what rate does the search light rotate when you are 15 meters from the point on the path closest to the search light?

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## 14. Related Rates and Angles

- You walk along a straight path at a speed of 4 m/s. A search light is located on the ground, a perpendicular distance of 20 m from your path. The light stays focused on you. At what rate does the search light rotate when you are 15 meters from the point on the path closest to the search light?
- So we need a relationship between the angle, the 20 meters and your distance along the path ➔ use the primary trig ratios to set this up ➔ the angle is that between the perpendicular (measuring 20 meters) and the path of the opposite side ➔ opposite and adjacent are related by the tangent ratio
- So $\tan(\theta) = x/20$ or $x = 20\tan(\theta)$
- Differentiating ➔ $d/dt\,(x) = d/dt\,(20\tan(\theta))$
- Thus $dx/dt = 20 \times \sec^2(\theta) \times d\theta/dt = 4$ m/s
- Then $d\theta/dt = 4 \div 20\sec^2(\theta)$

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## 14. Related Rates and Angles

- Then $d\theta/dt = 4 \div 20\sec^2(\theta)$
- To find $\sec^2(\theta)$, the measures in our triangle at the instant in question are the 20m as the perpendicular distance, 15m as the distance from the perpendicular, and then the hypotenuse as 25m ➔ so $\sec^2(\theta)$, $= (25/15)^2 = 25/16$
- Then $d\theta/dt = 4 \div (20 \times 25 \div 16) = 0.128$ rad/sec
- So the search light is rotating at 0.128 rad/sec

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## 15. Optimizing Volume



- ex 5. If 2700 $cm^2$ of material is available to make an open topped box with a square base, find the largest possible volume of the box.
- So once again, decide on what needs to be done
- Outline the steps of your strategy



## 15. Optimizing Volume



- ex 5. If 2700 $cm^2$ of material is available to make an open topped box with a square base, find the largest possible volume of the box.
- Start with what you want to "optimize" ➔ in this case the volume ➔ so we need a volume formula ➔ $V = L \times W \times H$
- We know that the base is a square (so $L = W$) ➔ so we can modify our formula as $V = L \times L \times H = L^2 \times H$
- But we are also given info. about the surface area (the material available to us), so we will work with a second relationship
- The surface area of a square based, on top box would be $SA = L \times L + 4L \times H$
- Thus 2700 $cm^2 = L^2 + LH$ then $H = (2700 - L^2)/L$
- So now we can finalize our formula, before differentiating as:
- $V(L) = L^2 \times (2700 - L^2)/L = L \times (2700 - L^2) = 2700L - L^3$
- But now let's consider the domain of our function ➔ $(0, \sqrt{2700})$