Lesson 42 – Implicit Differentiation

Calculus – Santowski

Lesson Objectives

- 1. Define the terms explicit and implicit equations
- 2. Implicitly differentiate implicitly defined equations
- 3. Determine the equation of tangents and normals of implicitly defined equations
- 4. Apply implicit differentiation to real world problems

Fast Five

- 1. Isolate *y* from $x^2 + y^2 = 25$
- 2. Isolate *y* from $3x - 2y + 10 = 0$
- 3. Isolate *y* from $y^2 - 4x + 7 = 0$
- 4. Isolate *y* from $3x^2 - 2y^3 = 1$
- 5. Isolate y from $2x^5 + x^4y + y^5 = 36$
- 6. Differentiate $36 = 2x^5 + x^4y + y^5$ on Wolframalpha

(A) Review

- Up to this point, we have always defined functions by expressing one variable *explicitly* in terms of another i.e. $y = f(x) = x^2 - 1/x + x$
- In other courses, we have also seen functions expressed *implicitly* i.e. in terms of both variables i.e. $x^2 + y^2 = 25$
- In simple implicit functions, we can always isolate the *y* term to rewrite the equation in *explicit* terms
  - i.e. $y = \pm\sqrt{(25 - x^2)}$
- In other cases, rewriting an implicit relation is not so easy i.e. $2x^5 + x^4y + y^5 = 36$

(A) Review

- Differentiate the following (d/dx):

$$(i)\ \left(5x^3 - 7x + 1\right)^5$$
$$(ii)\ \left[f(x)\right]^5$$
$$(iii)\ \left[y(x)\right]^5$$
$$(iv)\ y^5$$

(A) Review

- We need to agree on one convention ➔ when we see a y term in an implicitly (or explicitly defined equation), we will understand that we are really saying y(x) ➔ i.e. that y(x) is a differentiable function in *x*
- Therefore, if we see y^5, then we will interpret this expression as $(y(x))^5$ ➔ it is therefore differentiable in x.

(B) Derivatives

- We apply the chain rule in that we can recognize $[y(x)]^5$ as a composed function with the "inner" function being y(x) and the "outer" function is x^5
- So then according to the chain rule,

$$= \frac{d}{d(y(x))}\left(y(x)\right)^5 \times \frac{d}{dx}\, y(x)$$
$$= 5\left(y(x)\right)^4 \times \frac{d(y(x))}{dx}$$
$$= 5y^4 \times \frac{dy}{dx}$$

(C) Implicit Differentiation

- Example: Find dy/dx for $2x^5 + x^4y + y^5 = 36$
- There are two strategies that must be used
- First, the basic rule of equations is that we can do anything to an equation, provided that we do the same thing to both sides of the equation.
- So it will relate to taking derivatives ➔ we will start by taking the derivative of both sides.
- Secondly, then, work with the various implicitly defined fcns .....

(C) Implicit Differentiation

$$2x^5 + x^4y + y^5 = 36$$
$$\frac{d}{dx}\left(2x^5\right) + \frac{d}{dx}\left(x^4y\right) + \frac{d}{dx}\left(y^5\right) = \frac{d}{dx}\left(36\right)$$
$$10x^4 + \left(\left(4x^3 \times y\right) + \left(x^4 \times \frac{dy}{dx}\right)\right) + 5y^4 \times \frac{dy}{dx} = 0$$
$$10x^4 + 4x^3y + x^4\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0$$
$$10x^4 + 4x^3y = (-x^4 - 5y^4)\frac{dy}{dx}$$
$$\frac{10x^4 + 4x^3y}{-x^4 - 5y^4} = \frac{dy}{dx}$$

(D) In Class Examples

- ex 2. Find dy/dx if $x + \sqrt{y} = x^2y^3 + 5$
- ex 3. Find the slope of the tangent line drawn to $x^2 + 2xy + 3y^2 = 27$ at $x = 0$.
- ex 4. Determine the equation of the tangent line to the ellipse $4x^2 + y^2 - 8x + 6y = 12$ at $x = 3$.
- ex 5. Find d^2y/dx^2 of $x^3 + y^3 = 6$

Fast Five Revisited

- Differentiate Implicitly:
- 1. $x^2 + y^2 = 25$
- 2. $3x - 2y + 10 = 0$
- 3. $y^2 - 4x + 7 = 0$
- 4. $3x^2 - 2y^3 = 1$

$$e^{2x+3y} = x^2 - \ln\left(xy^3\right)$$

$$e^{xy} = 2x + y$$

(D) In Class Example

- (1) Given the ellipse $x^2 - xy + y^2 = 3$
  - (a) Determine the equation of the line normal to the curve at the point (−1,1)
  - (b) Determine a second point at which this normal line intersects the ellipse
  - (c) Graph on DESMOS to check your answer
- (2) Find all points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent is −1.

"Level 67" Level Questions

- 1. Find the equation of the lines that are tangent to the ellipse $x^2 + 4y^2 = 16$ AND also pass through the point (4,6)
- 2. Prove that the curves defined by $x^2 - y^2 = k$ and $xy = p$ intersect orthogonally for all values of of the constants k and p. Illustrate with a sketch

"Level 67" Level Questions

- Find the equation of the tangent line at the point (a,b) on the curve $x^{2/3} + y^{2/3} = 1$. Hence, show that the distance between the x- and y-intercepts of the tangent line is independent of the point of tangency

Resources

- You can watch the following ppt/pdfs:
- http://mrsantowski.tripod.com/2014MathHL/Resources/Implicit_Diff_Part_1.pdf
- http://mrsantowski.tripod.com/2014MathHL/Resources/Implicit_Diff_Part_2.pdf
- https://www.youtube.com/watch?v=anq_8ARu08g
- https://www.youtube.com/watch?v=rtZqpBSowJU
- https://www.youtube.com/watch?v=1scXr6g7HdA