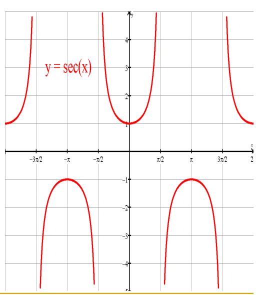
Lesson 41 – Derivatives of Secondary Trig Functions & Inverse Trig Functions

IB Math HL - Santowski

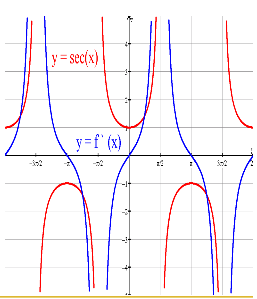
(A) Derivatives of f(x) = sec(x) - Graphically

- For y = sec(x) on $(-2\pi,2\pi)$
- Fcn is con up on $(-2\pi,-3\pi/2)$, $(-\pi/2, \pi/2)$, $(3\pi/2,2\pi)$
- Fcn is con down elsewhere
- Fcn has max at $-\pi$, π
- Fcn has min at $-2\pi,0,2\pi$
- Fcn increases on $(-2\pi,-3\pi/2)$, $(-3\pi/2,-\pi)$, $(0,\pi/2)$, $(\pi/2,\pi)$
- Fcn decreases elsewhere
- Fcn has asymptotes at $\pm 3\pi/2$, $\pm\pi/2$



(A) Derivatives of f(x) = sec(x) - Graphically

- For y = sec(x) on $(-2\pi,2\pi)$
- Fcn is con up on $(-2\pi,-3\pi/2)$, $(-\pi/2, \pi/2)$, $(3\pi/2,2\pi)$ ➔ ∴ f \` increases here
- Fcn is con down elsewhere ➔ ∴ f \` decreases here
- Fcn has max at $-\pi$, π ➔ roots on f \`
- Fcn has min at $-2\pi,0,2\pi$ ➔ roots on f \`
- Fcn increases on $(-2\pi,-3\pi/2)$, $(-3\pi/2,-\pi)$, $(0,\pi/2)$, $(\pi/2,\pi)$ ➔ f \` is positive
- Fcn decreases elsewhere ➔ f \` is negative
- Fcn has asymptotes at $\pm 3\pi/2$, $\pm\pi/2$ ➔ f \` has asymptotes



(B) Derivative of f(x) = sec(x) - Algebraically

- We will use the fact that sec(x) = 1/cos(x) to find the derivative of sec(x)

$$\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right)$$
$$\frac{d}{dx}(\sec(x)) = \frac{\frac{d}{dx}(1)\times\cos(x) - \frac{d}{dx}(\cos(x))\times 1}{\cos^2 x}$$
$$\frac{d}{dx}(\sec(x)) = \frac{(0)\times\cos x + \sin x}{\cos^2 x}$$
$$\frac{d}{dx}(\sec(x)) = \frac{\sin x}{\cos^2 x} = \frac{1\times\sin x}{\cos x\times\cos x}$$
$$\frac{d}{dx}(\sec(x)) = \frac{1}{\cos x}\times\frac{\sin x}{\cos x}$$
$$\frac{d}{dx}(\sec(x)) = \sec x\times\tan x$$

(C) Derivatives of f(x) = csc(x) and f(x) = cot(x)

- We can run through a similar curve analysis and derivative calculations to find the derivatives of the cosecant and cotangent functions as well
- The derivatives turn out to be as follows:
- d/dx csc(x) = -csc(x) cot(x)
- d/dx cot(x) = -csc^2(x)

(D) Summary of Trig Derivatives

- primary trig fcns:

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

- secondary trig fcns:

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$
$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

(E) Examples

- (i) Differentiate $f(x) = \dfrac{1}{1+\tan(x)}$
- (ii) Differentiate $h(x) = 2\csc^2(3x^2)$
- (iii) find dy/dx if tan(y) = x^2
- (iv) find the slope of the tangent line to y = tan(csc(x)) when sin(x) = 1/π on the interval (0,π/2)

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(E) Examples

- (i) Differentiate $f(x) = \dfrac{\tan(x)-1}{\sec(x)}$
- (ii) Differentiate *h*(*x*) = 2*x* + cot(*x*). Hence, find the x coordinates of the (i) minimums and (ii) maximums
- (iii) find the equation of the tangent line to y = sec(x) – 2cos(x) at the point (π/3,1)
- (iv) Find the intervals of concavity of y = sec(x) + tan(x)

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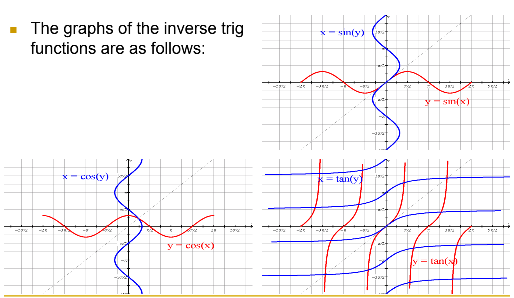
(A) Graphs of Inverse Trig Functions

- The graphs of the inverse trig functions are as follows:



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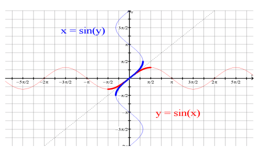
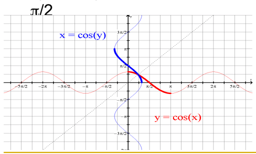
(B) Inverse Trig as Functions – Restrictions

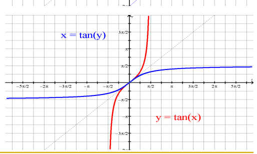
- From the graphs previously shown, the inverse trig "relations" are not functions since the domain elements do not "match" the range elements i.e. ➔ not one-to-one
- So we need to make domain restrictions in the original function such that when we "invert", our inverse does turn out to be a function
- What domain restrictions shall we make??

(B) Inverse Trig as Functions – Restrictions

- For y = sin(x) ➔between a max and min ($-\pi/2$ and $\pi/2$)
- For y = cos(x) ➔ between a max and min (0 and π)
- For y = tan(x) ➔ use one cycle, say between $-\pi/2$ and $\pi/2$







(C) Derivative of f(x) = $\sin^{-1}(x)$ on $(-\tfrac{1}{2}\pi,\tfrac{1}{2}\pi)$

- If y = sin(x), then to make the inverse, x = sin(y) and we can use implicit differentiation to find dy/dx

$$\frac{d}{dx}(x)=\frac{d}{dx}(\sin(y))$$
$$1=\frac{d}{dy}(\sin(y))\times\frac{dy}{dx}$$
$$1=\cos(y)\times\frac{dy}{dx}$$
$$\frac{1}{\cos(y)}=\frac{dy}{dx}$$

- But can we make a substitution for cos(y)??

$$\sin^2 y+\cos^2 y=1$$
$$\cos y=\sqrt{1-\sin^2 y}\Leftrightarrow x=\sin y$$
$$\cos y=\sqrt{1-x^2}$$
$$\Updownarrow$$
$$\frac{d}{dx}\left(\sin^{-1}(x)\right)=\frac{1}{\cos y}$$
$$\frac{d}{dx}\left(\sin^{-1}(x)\right)=\frac{1}{\sqrt{1-x^2}}$$

(D) Derivative of $f(x) = \cos^{-1}(x)$ on $(0,\pi)$

- If y = cos(x), then to make the inverse, x = cos(y) and we can use implicit differentiation to find dy/dx

$$\frac{d}{dx}(x) = \frac{d}{dx}(\cos(y))$$
$$1 = \frac{d}{dy}(\cos(y)) \times \frac{dy}{dx}$$
$$1 = -\sin(y) \times \frac{dy}{dx}$$
$$-\frac{1}{\sin(y)} = \frac{dy}{dx}$$

- But can we make a substitution for sin(y)??
-

$$\sin^2 y + \cos^2 y = 1$$
$$\sin y = \sqrt{1-\cos^2 y} \Leftrightarrow x = \cos y$$
$$\sin y = \sqrt{1-x^2}$$
$$\Updownarrow$$
$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sin y}$$
$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$$

(E) Derivative of $f(x) = \tan^{-1}(x)$ on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$

- If y = tan(x), then to make the inverse, x = tan(y) and we can use implicit differentiation to find dy/dx

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan(y))$$
$$1 = \frac{d}{dy}(\tan(y)) \times \frac{dy}{dx}$$
$$1 = \sec^2(y) \times \frac{dy}{dx}$$
$$\frac{1}{\sec^2(y)} = \frac{dy}{dx}$$

- But can we make a substitution for $\sec^2(y)$??

$$\sec^2 y = 1 + \tan^2 y$$
$$\sec^2 y = 1 + (\tan y)^2 \Leftrightarrow x = \tan y$$
$$\sec^2 y = 1 + x^2$$
$$\Updownarrow$$
$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{\sec^2 y}$$
$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$

(F) Summary of Trig Inverse Derivatives

- The three derivatives of the inverse of the trig. primary functions are:

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$

(F) Summary of Trig Inverse Derivatives

- Given the three derivatives of the inverse of the trig. primary functions, determine the derivatives of :

$$\frac{d}{dx}\left(\sin^{-1}(kx)\right)$$
$$\frac{d}{dx}\left(\cos^{-1}(kx)\right)$$
$$\frac{d}{dx}\left(\tan^{-1}(kx)\right)$$

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(H) Examples

- Differentiate y = $\sin^{-1}(1-x^2)$
- Differentiate $f(x) = x\tan^{-1}\sqrt{x}$
- Differentiate y = $\cos^{-1}(\sin(x))$
- If y = $\tan^{-1}(x/y)$, find dy/dx

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(H) Examples

- Problems and Solutions to Differentiation of Inverse Trigonometric Functions from UC Davis

$$\text{(i)}\ \ f(x) = x^2\arcsin(x)$$
$$\text{(ii)}\ \ g(x) = x(\tan^{-1}x)^2$$
$$\text{(iii)}\ h(x) = \arctan(e^{-x^2})$$
$$\text{(iv)}\ y = \frac{1+\tan^{-1}x}{2-3\tan^{-1}x}$$

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(H) Examples

- Problems and Solutions to Differentiation of Inverse Trigonometric Functions from UC Davis
- (i) Find the equation of the tangent to the function
  y = arctan(ln$x$) at x = e.
- (ii) Given that y = $\sin^{-1}(x) + \cos^{-1}(x)$. Show that:
  - (a) y'(x) = 0
  - (b) $\sin^{-1}(x) + \cos^{-1}(x)$ = pi/2