Lesson 39 - Derivatives of Transcendental Functions

IB Math HL - Santowski

Opening Exercises #1

Differentiate the following functions with respect to the appropriate variable.

(a) $s = 12t^4 - \sqrt{t}$ (b) $Q = \left(n + \frac{1}{n^2}\right)^2$ (c) $P = \sqrt{r}(r + \sqrt[3]{r} - 2)$

(d) $T = \frac{(\theta - \sqrt{\theta})^3}{\theta}$ (e) $A = 40L - L^3$ (f) $F = \frac{50}{v^2} - v$

(g) $V = 2l^3 + 5l$ (h) $A = 2\pi h + 4h^2$ (i) $N = n^4 - \sqrt[3]{n} + \pi n$

Differentiate the following with respect to the independent variable.

(a) $v = \frac{2}{3}\left(5 - \frac{2}{t^2}\right)$ (b) $S = \pi r^2 + \frac{20}{r}$ (c) $q = \sqrt{s^5} - \frac{3}{s}$

(d) $h = \frac{2 - t + t^2}{t^3}$ (e) $L = \frac{4 - \sqrt{b}}{b}$ (f) $W = (m-2)^2(m+2)$

Opening Exercises #2

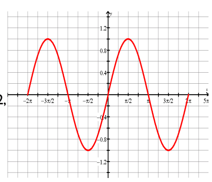
- Prepare a graph of the following functions and then prepare a sketch of their derivatives:
- (a) y = sin(x)
- (b) y = cos(x)
- (c) y = tan(x)
- (d) y = e^x
- (e) y = ln(x)

Lesson Objectives

- (1) Work with basic strategies for developing new knowledge in Mathematics ➔ (a) graphical, (b) technology, (c) algebraic
- (2) Introduce & work with fundamental trig limits
- (3) Determine the derivative of trigonometric functions
- (4) Apply & work with the derivatives of the trig functions

(A) Derivative of the Sine Function - Graphically

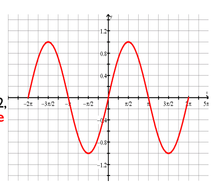
- We will predict the derivative of f(x) = sin(x) from a GRAPHICAL ANALYSIS perspective:
- We will simply sketch 2 cycles
- (i) we see a maximum at $\pi/2$ and -3 $\pi/2$ ➔ derivative must have ……..? ?
- (ii) we see a minimum at -$\pi/2$ and 3 $\pi/2$ ➔ derivative must have ……..? ?
- (iii) we see intervals of increase on (-2π,-3π/2), (-π/2, π/2), (3π/2,2π) ➔ derivative must …….?
- (iv) the opposite is true of intervals of decrease
- (v) intervals of concave up are (-π,0) and (π ,2π) ➔ so derivative must …….? ?
- (vi) the opposite is true for intervals of concave up



- So the derivative function must look like ➔ ??

(A) Derivative of the Sine Function - Graphically

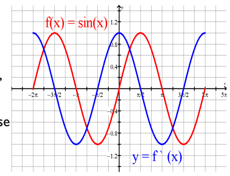
- We will predict the derivative of f(x) = sin(x) from a GRAPHICAL ANALYSIS perspective:
- We will simply sketch 2 cycles
- (i) we see a maximum at $\pi/2$ and -3 $\pi/2$ ➔ derivative must have **ZEROES here**
- (ii) we see a minimum at -$\pi/2$ and 3 $\pi/2$ ➔ derivative must have **ZEROES here**
- (iii) we see intervals of increase on (-2π,-3π/2), (-π/2, π/2), (3π/2,2π) ➔ derivative must be **positive here**
- (iv) the opposite is true of intervals of decrease
- (v) intervals of concave up are (-π,0) and (π ,2π) ➔ so derivative must be **increasing here**
- (vi) the opposite is true for intervals of concave up



- So the derivative function must look like ➔ **cosine graph**

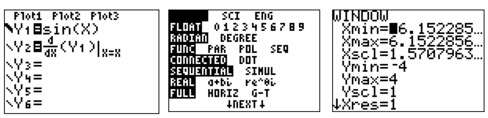
(A) Derivative of the Sine Function - Graphically

- We will predict the derivative of f(x) = sin(x) from a GRAPHICAL ANALYSIS perspective:
- We will simply sketch 2 cycles
- (i) we see a maximum at $\pi/2$ and $-3\pi/2$ ➔ derivative must have x-intercepts
- (ii) we see intervals of increase on $(-2\pi,-3\pi/2)$, $(-\pi/2, \pi/2)$, $(3\pi/2,2\pi)$ ➔ derivative must be positive on these intervals
- (iii) the opposite is true of intervals of decrease
- (iv) intervals of concave up are $(-\pi,0)$ and $(\pi, 2\pi)$ ➔ so derivative must increase on these domains
- (v) the opposite is true for intervals of concave up
- So the derivative function must look like ➔ the cosine function!!



(A) Derivative of the Sine Function - Technology

- We will predict the what the derivative function of f(x) = sin(x) looks like from our graphing calculator:



(A) Derivative of the Sine Function - Technology

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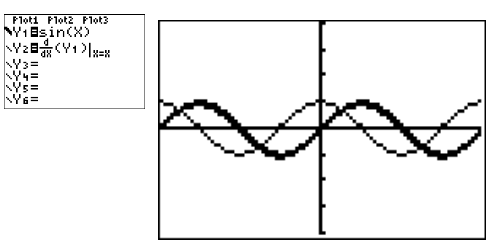
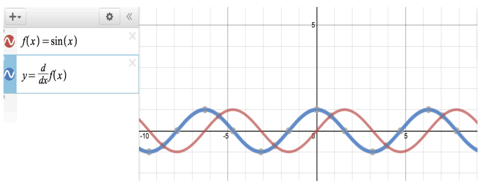


(A) Derivative of the Sine Function - Technology

- We will predict the what the derivative function of f(x) = sin(x) looks like from DESMOs:



(B) Derivative of Sine Function - Algebraically

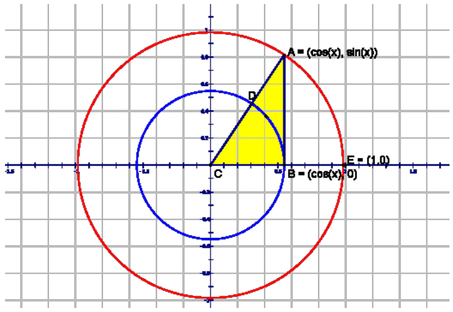
- We will go back to our limit concepts for an algebraic determination of the derivative of y = sin(x)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$
$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}$$
$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x)[\cos(h)-1)]+\sin(h)\cos(x)}{h}$$
$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x)[\cos(h)-1]}{h}+\lim_{h\to 0}\frac{\sin(h)\cos(x)}{h}$$
$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}(\sin(x))\times\lim_{h\to 0}\frac{\cos(h)-1}{h}+\lim_{h\to 0}\frac{\sin(h)}{h}\times\lim_{h\to 0}\cos(x)$$
$$\frac{d}{dx}\sin(x) = \sin(x)\times\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\times\lim_{h\to 0}\frac{\sin(h)}{h}$$

(B) Derivative of Sine Function - Algebraically

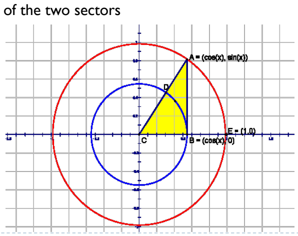
- So we come across 2 special trigonometric limits:
- $\lim_{h\to 0}\frac{\sin(h)}{h}$ and $\lim_{h\to 0}\frac{\cos(h)-1}{h}$
- So what do these limits equal?
- Since we are looking at these ideas from an ALGEBRAIC PERSPECTIVE ➔ We will introduce a new theorem called a Squeeze (or sandwich) theorem ➔ if we that our limit in question lies between two known values, then we can somehow "squeeze" the value of the limit by adjusting/ manipulating our two known values

(C) Applying "Squeeze Theorem" to Trig. Limits



(C) Applying "Squeeze Theorem" to Trig. Limits

- We have sector DCB and sector ACE "squeezing" the triangle ACB
- So the area of the triangle ACB should be "squeezed between" the area of the two sectors



(C) Applying "Squeeze Theorem" to Trig. Limits

- Working with our area relationships (make h = θ)

$$\tfrac{1}{2}(CB)^2(\theta) \le \tfrac{1}{2}(CB)(CA) \le \tfrac{1}{2}(CA)^2(\theta)$$
$$\tfrac{1}{2}\theta \times \cos^2(\theta) \le \tfrac{1}{2}\sin(\theta)\cos(\theta) \le \tfrac{1}{2}\theta \times (1)^2$$
$$\theta\cos^2(\theta) \le \sin(\theta)\cos(\theta) \le \theta$$
$$\frac{\theta\cos^2(\theta)}{\theta\cos(\theta)} \le \frac{\sin(\theta)\cos(\theta)}{\theta\cos(\theta)} \le \frac{\theta}{\theta\cos(\theta)}$$
$$\cos(\theta) \le \frac{\sin(\theta)}{\theta} \le \frac{1}{\cos(\theta)}$$

- We can "squeeze or sandwich" our ratio of sin(h)/h between cos(h) and 1/cos(h)

(C) Applying "Squeeze Theorem" to Trig. Limits

- Now, let's apply the squeeze theorem as we take our limits as h →0⁺ (and since sin(h) has even symmetry, the LHL as h →0⁻)

$$\lim_{h\to 0}\cos(h) \le \lim_{h\to 0}\frac{\sin(h)}{h} \le \lim_{h\to 0}\frac{1}{\cos(h)}$$

$$1 \le \lim_{h\to 0}\frac{\sin(h)}{h} \le 1$$

$$\therefore \lim_{h\to 0}\frac{\sin(h)}{h} = 1$$

- Follow the link to Visual Calculus - Trig Limits of sin(h)/h to see their development of this fundamental trig limit

(C) Applying "Squeeze Theorem" to Trig. Limits

- Now what about (cos(h) – 1) / h and its limit ➔ we will treat this algebraically

$$\lim_{h\to 0}\frac{\cos(h)-1}{h}$$

$$= \lim_{h\to 0}\frac{(\cos(h)-1)(\cos(h)+1)}{h(\cos(h)+1)}$$

$$= \lim_{h\to 0}\frac{\cos^2(h)-1}{h(\cos(h)+1)}$$

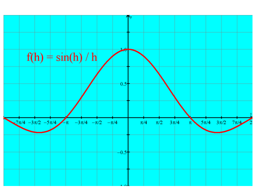
$$= \lim_{h\to 0}\frac{-\sin^2(h)}{h(\cos(h)+1)}$$

$$= -1\times\lim_{h\to 0}\frac{\sin(h)}{h}\times\lim_{h\to 0}\frac{\sin(h)}{(\cos(h)+1)}$$

$$= -1\times 1\times\left(\frac{0}{1+1}\right)$$

$$= 0$$

(D) Fundamental Trig. Limits ➔ Graphic and Numeric Verification



x	y
-0.05000	0.99958
-0.04167	0.99971
-0.03333	0.99981
-0.02500	0.99990
-0.01667	0.99995
-0.00833	0.99999
0.00000	undefined
0.00833	0.99999
0.01667	0.99995
0.02500	0.99990
0.03333	0.99981
0.04167	0.99971
0.05000	0.99958

(D) Derivative of Sine Function

- Since we have our two fundamental trig limits, we can now go back and algebraically verify our graphic "estimate" of the derivative of the sine function:

$$\lim_{h\to 0}\frac{\sin(h)}{h}=1$$

$$\lim_{h\to 0}\frac{\cos(h)-1}{h}=0$$

$$\frac{d}{dx}(\sin(x))=\sin(x)\times\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\times\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$\frac{d}{dx}(\sin(x))=\sin(x)\times 0+\cos(x)\times 1$$

$$\frac{d}{dx}(\sin(x))=\cos(x)$$

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(E) Derivative of the Cosine Function

- Knowing the derivative of the sine function, we can develop the formula for the cosine function
- First, consider the graphic approach as we did previously

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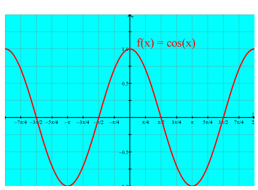
(E) Derivative of the Cosine Function

- We will predict the what the derivative function of f(x) = cos(x) looks like from our curve sketching ideas:
- We will simply sketch 2 cycles
- (i) we see a maximum at 0, -2π & 2π ➔ derivative must have x-intercepts
- (ii) we see intervals of increase on (-π,0), (π, 2π) ➔ derivative must increase on this intervals
- (iii) the opposite is true of intervals of decrease
- (iv) intervals of concave up are (-3π/2,-π/2) and (π/2 ,3π/2) ➔ so derivative must increase on these domains
- (v) the opposite is true for intervals of concave up
- So the derivative function must look like ➔ some variation of the sine function!!



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(E) Derivative of the Cosine Function

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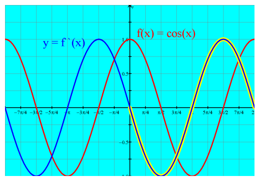
- (iv) intervals of concave up are (-3π/2,-π/2) and (π/2 ,3π/2) ➔ so derivative must increase on these domains
- (v) the opposite is true for intervals of concave up

- So the derivative function must look like ➔ the negative sine function!!



(E) Derivative of the Cosine Function

- Knowing the derivative of the sine function, we can develop the formula for the cosine function
- First, consider the algebraic approach as we did previously
- Recalling our IDENTITIES ➔ cos(x) can be rewritten in TERMS OF SIN(X) as:
- (a) y = sin(pi/2 – x)
- (b) y = sqrt(1 – $\sin^2(x)$)

(E) Derivative of the Cosine Function

- Let's set it up algebraically:

$$\frac{d}{dx}(\cos(x)) = \frac{d}{dx}\left(\sin\left(\frac{\pi}{2}-x\right)\right)$$

$$\frac{d}{dx}(\cos(x)) = \frac{d}{d\left(\frac{\pi}{2}-x\right)}\left(\sin\left(\frac{\pi}{2}-x\right)\right)\times\frac{d}{dx}\left(\frac{\pi}{2}-x\right)$$

$$\frac{d}{dx}(\cos(x)) = \cos\left(\frac{\pi}{2}-x\right)\times(-1)$$

$$\frac{d}{dx}(\cos(x)) = \sin(x)\times -1 = -\sin(x)$$

(E) Derivative of the Cosine Function

- Let's set it up algebraically:

$$\frac{d}{dx}(\cos(x)) = \frac{d}{dx}\left(\sqrt{1-\sin^2(x)}\right)$$

$$\frac{d}{dx}(\cos(x)) = \frac{1}{2}\left(1-\sin^2(x)\right)^{-\frac{1}{2}} \bullet -2\sin(x)\cos(x)$$

$$\frac{d}{dx}(\cos(x)) = \frac{1}{2\sqrt{1-\sin^2(x)}} \bullet -2\sin(x)\cos(x)$$

$$\frac{d}{dx}(\cos(x)) = \frac{-2\sin(x)\cos(x)}{2\sqrt{1-\sin^2(x)}}$$

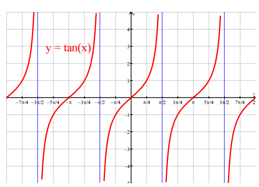
$$\frac{d}{dx}(\cos(x)) = \frac{-2\sin(x)\cos(x)}{2\sqrt{\cos^2(x)}}$$

$$\frac{d}{dx}(\cos(x)) = \frac{-2\sin(x)\cos(x)}{2\cos(x)}$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

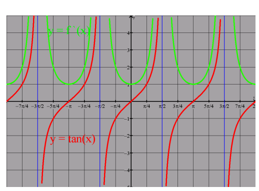
(F) Derivative of the Tangent Function - Graphically

- So we will go through our curve analysis again
- f(x) is constantly increasing within its domain
- f(x) has no max/min points
- f(x) changes concavity from con down to con up at 0,$\pm\pi$
- f(x) has asymptotes at $\pm 3\pi$
- /2, $\pm\pi$/2



(F) Derivative of the Tangent Function - Graphically

- So we will go through our curve analysis again:
- F(x) is constantly increasing within its domain → f '(x) should be positive within its domain
- F(x) has no max/min points → f '(x) should not have roots
- F(x) changes concavity from con down to con up at 0,$\pm\pi$ → f '(x) changes from decrease to increase and will have a min
- F(x) has asymptotes at $\pm 3\pi$ →
- /2, $\pm\pi$/2 → derivative should have asymptotes at the same points



(F) Derivative of the Tangent Function - Algebraically

- We will use the fact that tan(x) = sin(x)/cos(x) to find the derivative of tan(x)

$$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

$$\frac{d}{dx}(\tan(x)) = \frac{\frac{d}{dx}(\sin(x)) \times \cos(x) - \frac{d}{dx}(\cos(x)) \times \sin(x)}{(\cos(x))^2}$$

$$\frac{d}{dx}(\tan(x)) = \frac{\cos(x) \times \cos(x) - (-\sin(x)) \times \sin(x)}{\cos^2 x}$$

$$\frac{d}{dx}(\tan(x)) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$\frac{d}{dx}(\tan(x)) = \frac{1}{\cos^2 x} = \sec^2 x$$

DERIVATIVES OF EXPONENTIAL FUNCTIONS

CALCULUS - SANTOWSKI

(A) EXPLORATION – PART 1

- 1. Sketch $y = b^x$ Each partner at the table will use a different value for *b*
- 2. PREDICT the features of the graph of its derivative by answering the following Q
- (a) Identify the intervals of increase and decrease
- (b) identify the critical values
- (c) From this information (and knowing what each means about the derivative), prepare a hand drawn graph of the derivative

(B) EXPLORATION – PART 2

- We will go back to our "first principles" - that being the idea that we can determine instantaneous rates of changes using tangent lines
- (1) Use GDC to draw the tangent lines at various x values
- (2) Record the slopes of the tangent lines on a table.
- (3) Prepare a scatter plot from the table of values.
- (4) Describe scatter plot

EXPLORATION – PART 3

- Now let's use graphing technology:
- Use the TI-84 to prepare a graph of the derivative of $y = b^x$.
- What is the derivative of $y = b^x$?
- Confirm that your equation for the derivative is correct (and show/explain how you confirmed this.)

EXPLORATION – PART 4

- Now we will use algebra to PROVE that our observations were correct.
- So we go back to our limit definition of a derivative:
- Our definition is: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- So work with it ……

DERIVATIVE OF EXPONENTIAL FUNCTIONS

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{b^{x+h}-b^x}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{b^x b^h-b^x}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{b^x(b^h-1)}{h}$$

$$f'(x) = \lim_{h\to 0}\left(b^x\right)\times\lim_{h\to 0}\frac{b^h-1}{h}$$

$$f'(x) = b^x\times\lim_{h\to 0}\frac{b^h-1}{h}$$

(B) Investigating the Limits

- Investigate lim $_{h\to 0}$ $(2^h - 1)/h$ numerically with a table of values

x	y
-0.00010	0.69312
-0.00007	0.69313
-0.00003	0.69314
0.00000	undefined
0.00003	0.69316
0.00007	0.69316
0.00010	0.69317

- And we see the value of 0.693 as an approximation of the limit

- Investigate lim $_{h\to 0}$ $(3^h - 1)/h$ numerically with a table of values

x	y
-0.00010	1.09855
-0.00007	1.09857
-0.00003	1.09859
0.00000	undefined
0.00003	1.09863
0.00007	1.09865
0.00010	1.09867

- And we see the value of 1.0986 as an approximation of the limit

(B) Investigating the Limits

- Investigate lim $h\to 0$ $(4^h - 1)/h$ numerically with a table of values

x	y
-0.00010	1.38620
-0.00007	1.38623
-0.00003	1.38626
0.00000	undefined
0.00003	1.38633
0.00007	1.38636
0.00010	1.38639

- And we see the value of 1.386 as an approximation of the limit

- Investigate lim $h\to 0$ $(e^h - 1)/h$ numerically with a table of values

x	y
-0.00010	0.99995
-0.00007	0.99997
-0.00003	0.99998
0.00000	undefined
0.00003	1.00002
0.00007	1.00003
0.00010	1.00005

- And we see the value of 1.000 as an approximation of the limit

(C) Special Limits - Summary

- Is there a pattern to these numbers ➔ the number 0.693 (coming from base 2), 1.0896 (coming from base = 3), 1.386 (base 4)
- To explore, we can rewrite a^x in base e as $e^{(\ln a)x}$
- So if d/dx e^x was e^x, then d/dx $e^{(\ln a)x}$ must be $e^{(\ln a)x}$ times lna (by the chain rule)
- And so: ln(2) = 0.693
- And so: ln(3) = 1.0986
- And so: ln(4) = 1.386

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(D) Derivatives of Exponential Functions - Summary

- The derivative of an exponential function was

$$\frac{d}{dx} a^x = a^x \times \lim_{h \to 0} \frac{a^h - 1}{h}$$

- Which we will now rewrite as

$$\frac{d}{dx} a^x = a^x \times \ln a$$

- And we will see one special derivative ➔ when the exponential base is e, then the derivative becomes:

$$\frac{d}{dx} e^x = e^x \times \ln e = e^x \times 1 = e^x$$

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