

Lesson 32 (Day 2) - Limits

Calculus - Mr Santowski

1/22/17

Mr. Santowski - Calculus

1

Lesson Objectives

- 1. Define limits
- 2. Use algebraic, graphic and numeric (AGN) methods to determine if a limit exists
- 3. Use algebraic, graphic and numeric methods to determine the value of a limit, if it exists
- 4. Use algebraic, graphic and numeric methods to determine the value of a limit at infinity, if it exists
- 5. Be able to state and then work with the various laws of limits
- 6. Apply limits to application/real world problems

1/22/17

Mr. Santowski - Calculus

2

Fast Five - Limits (Lesson #2)

- Consider the following limit(s):

$$\lim_{x \rightarrow \infty} f(x) \text{ if } f(x) = \frac{1}{x^n}$$

$$\lim_{x \rightarrow 0^+} f(x) \text{ if } f(x) = \frac{1}{x^n}$$

- Determine the limiting value(s) if possible.

1/22/17

Mr. Santowski - Calculus

3

(D) Limits at Infinity

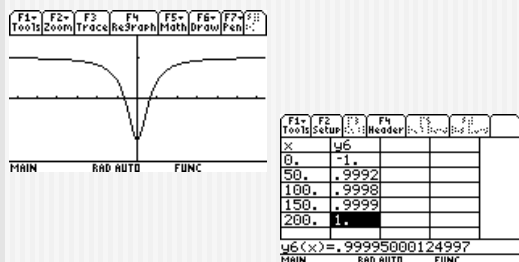
- In considering limits at infinity, we are being asked to make our x values infinitely large and thereby consider the "end behaviour" of a function
- Consider the limit $\lim_{x \rightarrow +\infty} \frac{x^2 - 1}{x^2 + 1}$ numerically, graphically and algebraically
- We can generate a table of values and a graph (see next slide)
- So here the function approaches a limiting value, as we make our x values sufficiently large \rightarrow we see that f(x) approaches a limiting value of 1 \rightarrow in other words, a horizontal asymptote

1/22/17

Mr. Santowski - Calculus

4

(D) Limits at Infinity – Graph & Table



1/22/17

Mr. Santowski - Calculus

5

(D) Limits at Infinity – Algebra

$$\begin{aligned} & \lim_{x \rightarrow \infty} \left(\frac{x^2 - 1}{x^2 + 1} \right) \\ &= \lim_{x \rightarrow \infty} \left(\frac{x^2 - 1}{x^2 + 1} \right) \\ &= \lim_{x \rightarrow \infty} \left(\frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x^2}} \right) \\ &= \frac{1 - 0}{1 + 0} \\ &= 1 \end{aligned}$$

- Divide through by the highest power of x
- Simplify
- Substitute $x = \infty \rightarrow 1/\infty \rightarrow 0$

1/22/17

Mr. Santowski - Calculus

6

(D) Examples of Limits at Infinity

- Work through the following examples graphically, numerically or algebraically
 - Work through the following examples graphically, numerically or algebraically
- (i) $\lim_{x \rightarrow \infty} \left(\frac{3x^2 - x - 2}{-5x^2 + 4x + 1} \right)$ $\lim_{x \rightarrow -\infty} (\tan^{-1}(x))$
- (ii) $\lim_{x \rightarrow \infty} \left(\frac{3x^4 - x - 2}{-5x^2 + 4x + 1} \right)$
- (iii) $\lim_{x \rightarrow \infty} \left(\frac{3x^2 - x - 2}{-5x^4 + 4x + 1} \right)$ $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 2} - x)$

1/22/17

Mr. Santowski - Calculus

7

Limits at Infinity – Horizontal Asymptotes

Definition: Horizontal Asymptote
 The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b$$

In the last example, we saw that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$. Using this limit and the properties of limits, we can find the limits of other functions as x approaches infinity.

Example: $\lim_{x \rightarrow \infty} \left(5 - \frac{2}{x^2} \right) =$

1/22/17

Mr. Santowski - Calculus

8

Limits of Infinity – Vertical Asymptotes

Definition: Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^+} f(x) = \pm\infty$$

Please NOTE: Infinity is NOT a number, and thus the limit FAILS to exist in both of these cases. If this seems confusing, then use the notation as $x \rightarrow a$ (from the right or left), then the function $f(x) \rightarrow \pm\infty$.

Example: Find the vertical asymptotes of $f(x)$. Describe the behavior of $f(x)$ to the left and right of each asymptote.

a) $f(x) = \frac{x^2 - 1}{2x + 4}$

b) $f(x) = \frac{1 - x}{2x^2 - 5x - 3}$

c) $f(x) = \frac{x - 2}{3x^2 - 5x - 2}$

1/22/17

Mr. Santowski - Calculus

9

(D) Examples of Limits at Infinity

3. For each of the following, find $\lim_{x \rightarrow \infty} f(x)$, $\lim_{x \rightarrow -\infty} f(x)$

a) $f(x) = \frac{x - 2}{2x^2 + 3x - 5}$

b) $f(x) = \frac{4x^3 - 2x + 1}{x^2 - 2x + 1}$

c) $f(x) = \frac{3x^2 - x + 5}{x^2 - 4}$

d) $f(x) = \frac{|x|}{x}$

1/22/17

Mr. Santowski - Calculus

10

(D) Examples of Limits of Infinity

4. For each of the following, identify all vertical asymptotes and find $\lim_{x \rightarrow a^-} f(x)$, $\lim_{x \rightarrow a^+} f(x)$, and $\lim_{x \rightarrow a} f(x)$, where a is the x -value of the asymptote.

a) $f(x) = \frac{1}{x - 3}$

b) $f(x) = \frac{1}{x^2 - 4x + 4}$

c) $f(x) = \frac{x^2 - 3x - 4}{x^2 - 16}$

1/22/17

Mr. Santowski - Calculus

11

(D) Examples of Limits of Infinity

5. For each of the following piecewise functions, find the limit as $x \rightarrow \infty$, $x \rightarrow -\infty$, $x \rightarrow 0^-$, and $x \rightarrow 0^+$

a) $g(x) = \begin{cases} \frac{1}{x} & x < 0 \\ \frac{2x - 3}{x + 1} & x \geq 0 \end{cases}$

b) $g(x) = \begin{cases} \frac{3x}{x + 1} & x \leq 0 \\ \frac{1}{x^2} & x > 0 \end{cases}$

1/22/17

Mr. Santowski - Calculus

12

(D) Examples of Limits with Infinity

7. Evaluate the following limits without a calculator.

a) $\lim_{x \rightarrow \infty} \left(\frac{2}{x} + 1 \right) \left(\frac{5x^2 - 1}{x^2} \right)$

b) $\lim_{n \rightarrow \infty} \left(\frac{3n^3 - 5n}{n^3 - 2n^2 + 1} \right)$

c) $\lim_{x \rightarrow \infty} e^{-x} \cos x$

d) $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 + 10x + 21}$

1/22/17

Mr. Santowski - Calculus

13

Curve Sketching & Limits – Ex 1

Example: Sketch the function that satisfies the stated conditions.

$$\lim_{x \rightarrow 1} f(x) = 2$$

$$\lim_{x \rightarrow 2} f(x) = \infty$$

$$\lim_{x \rightarrow 3} f(x) = \infty$$

$$\lim_{x \rightarrow \infty} f(x) = -1$$

$$\lim_{x \rightarrow -\infty} f(x) = 0$$

$$\lim_{x \rightarrow -2} f(x) = \infty$$

$$\lim_{x \rightarrow -2} f(x) = -\infty$$

1/22/17

Mr. Santowski - Calculus

14

Curve Sketching & Limits – Ex 2

Example: Sketch the function that satisfies the stated conditions.

$$\lim_{x \rightarrow 2} f(x) = -1$$

$$\lim_{x \rightarrow 4} f(x) = -\infty$$

$$\lim_{x \rightarrow 4} f(x) = \infty$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

$$\lim_{x \rightarrow -\infty} f(x) = 2$$

1/22/17

Mr. Santowski - Calculus

15

(E) Limit Laws

- The limit of a constant function is the constant
- The limit of a sum is the sum of the limits
- The limit of a difference is the difference of the limits
- The limit of a constant times a function is the constant times the limit of the function
- The limit of a product is the product of the limits
- The limit of a quotient is the quotient of the limits (if the limit of the denominator is not 0)
- The limit of a power is the power of the limit
- The limit of a root is the root of the limit

1/22/17

Mr. Santowski - Calculus

16

(E) Limit Laws

- Here is a summary of some important limits laws:
- (a) sum/difference rule $\rightarrow \lim [f(x) \pm g(x)] = \lim f(x) \pm \lim g(x)$
- (b) product rule $\rightarrow \lim [f(x) \times g(x)] = \lim f(x) \times \lim g(x)$
- (c) quotient rule $\rightarrow \lim [f(x) \div g(x)] = \lim f(x) \div \lim g(x)$
- (d) constant multiple rule $\rightarrow \lim [kf(x)] = k \times \lim f(x)$
- (e) constant rule $\rightarrow \lim (k) = k$

- These limits laws are easy to work with, especially when we have rather straight forward polynomial functions

1/22/17

Mr. Santowski - Calculus

17

(E) Limit Laws - Examples

- Find $\lim_{x \rightarrow 2} (3x^3 - 4x^2 + 11x - 5)$ using the limit laws
- $\lim_{x \rightarrow 2} (3x^3 - 4x^2 + 11x - 5)$
- $= 3 \lim_{x \rightarrow 2} (x^3) - 4 \lim_{x \rightarrow 2} (x^2) + 11 \lim_{x \rightarrow 2} (x) - \lim_{x \rightarrow 2} (5)$
- $= 3(8) - 4(4) + 11(2) - 5$ (using simple substitution or use GDC)
- $= 25$

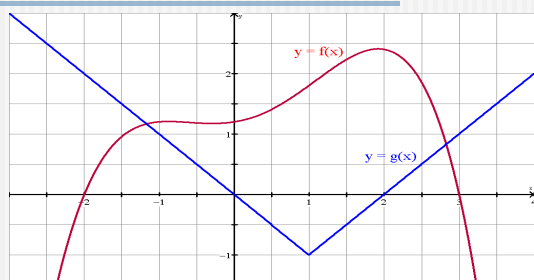
- For the rational function $f(x)$, find
- $\lim_{x \rightarrow 2} (2x^2 - x) / (0.5x^3 - x^2 + 1)$
- $= [2 \lim_{x \rightarrow 2} (x^2) - \lim_{x \rightarrow 2} (x)] / [0.5 \lim_{x \rightarrow 2} (x^3) - \lim_{x \rightarrow 2} (x^2) + \lim_{x \rightarrow 2} (1)]$
- $= (8 - 2) / (4 - 4 + 1)$
- $= 6$

1/22/17

Mr. Santowski - Calculus

18

(E) Limit Laws and Graphs



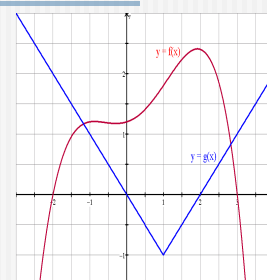
1/22/17

Mr. Santowski - Calculus

19

(E) Limit Laws and Graphs

- From the graph on this or the previous page, determine the following limits:
- (1) $\lim_{x \rightarrow 2} [f(x) + g(x)]$
- (2) $\lim_{x \rightarrow 2} [(f(x))^2 - g(x)]$
- (3) $\lim_{x \rightarrow 2} [f(x) \times g(x)]$
- (4) $\lim_{x \rightarrow 2} [f(x) \div g(x)]$
- (5) $\lim_{x \rightarrow 1} [f(x) + 5g(x)]$
- (6) $\lim_{x \rightarrow 1} [1/2f(x) \times (g(x))^2]$
- (7) $\lim_{x \rightarrow 2} [f(x) \div g(x)]$
- (8) $\lim_{x \rightarrow 2} [g(x) \div f(x)]$
- (9) $\lim_{x \rightarrow 3} [f(x) \div g(x)]$



1/22/17

Mr. Santowski - Calculus

20

(I) Exploration

- Research the DELTA-EPSILON definition of a limit
- Tell me what it is and be able to use it
- MAX 2 page hand written report (plus graphs plus algebra) + 2 Q quiz