Lesson 32 - Limits

Calculus - Mr Santowski

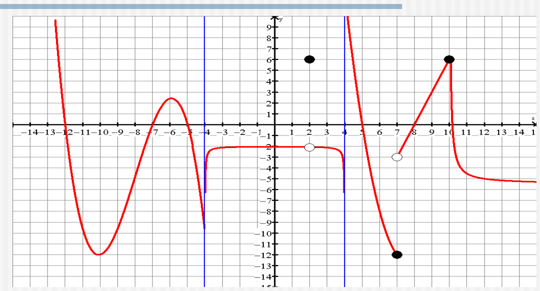
Lesson Objectives

- 1. Define limits
- 2. Use algebraic, graphic and numeric (AGN) methods to determine if a limit exists
- 3. Use algebraic, graphic and numeric methods to determine the value of a limit, if it exists
- 4. Use algebraic, graphic and numeric methods to determine the value of a limit at infinity, if it exists
- 5. Be able to state and then work with the various laws of limits
- 6. Apply limits to application/real world problems

Fast Five

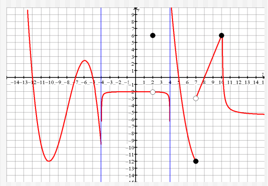
- You will get a hand out of the following three slides. You and your table have 5 minutes to answer the limit questions and then defend your answers (if challenged)

Fast Five - Limits and Graphs



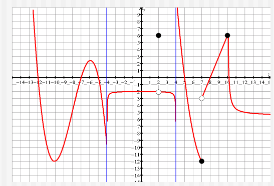
Fast Five - Limits and Graphs

- (A) Find function values at the following:
- (i) f(-10) =
- (ii) f(-4) =
- (iii) f(2) =
- (iv) f(7) =



Fast Five - Limits and Graphs

- Find the limit of the function f(x) at the following values:
- (i) the limit of f(x) at x = -10 is
- (ii) the limit of f(x) at x = -6 is
- (iii) the limit of f(x) at x = -4 is
- (iv) the limit of f(x) at x = 0 is
- (v) the limit of f(x) at x = 2 is
- (vi) the limit of f(x) at x = 4 is
- (vii) the limit of f(x) at x = 6 is
- (i) the limit of f(x) at x = 7 is
- (i) the limit of f(x) at x = 10 is



(A) Introduction to Limits

- Let *f* be a function and let *a* and *L* be real numbers. If
- 1. As *x* takes on values closer and closer (but not equal) to *a* on both sides of *a*, the corresponding values of *f*(*x*) get closer and closer (and perhaps equal) to *L*; and
- 2. The value of *f*(*x*) can be made as close to *L* as desired by taking values of *x* close enough to *a*;
- Then *L* is the LIMIT of *f*(*x*) as *x* approaches *a*
- Written as $\lim_{x \to a} f(x) = L$

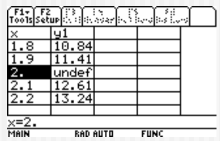
(A) Introduction to Limits

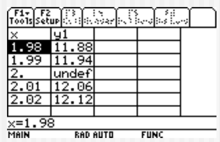
- We will work with $g(x) = \dfrac{x^3 - 8}{x - 2}$ and consider the function behaviour at x = 2
- We can express this idea of function behaviour at a point using limit notation as

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$$

(A) Introduction to Limits

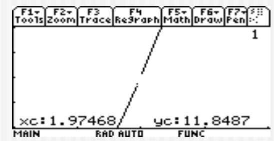
- We will explore the limit in a variety of ways: first using a ToV
- So notice what happens to the function values as x gets closer to 2 from both sides (RS 2.01, 2.02 & LS 1.98, 1.99)
- So we can predict a limiting function value of 12

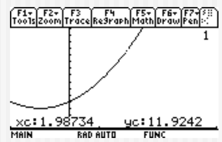


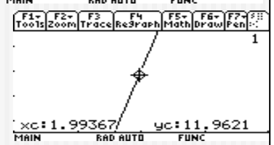


(A) Introduction to Limits

- We will explore the limit in a variety of ways: now using a graph and tracing the function values

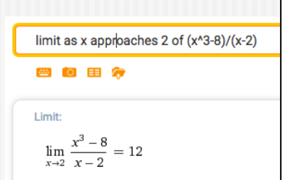






(A) Introduction to Limits

- We can use Wolframalpha to actually calculate the limit value for us
- So we have the confirmation of the limiting function value of 12 as we had previously with the table and the graph



(B) Determining Values of Limits

- Now, how does all the algebra tie into limits?
- If we try a direct substitution to evaluate the limit value, we get 0/0 which is indeterminate

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{(2)^3 - 8}{(2) - 2} = \frac{0}{0}$$

(B) Determining Values of Limits

- Is there some way that we can use our algebra skills to come to the same answer?
- Four skills become important initially: (1) factoring & simplifying, (2) rationalizing and (3) common denominators and (4) basic function knowledge

(B) Determining Values of Limits

- Consider the expression $g(x) = \frac{x^3 - 8}{x - 2}$
- Now can we factor a difference of cubes?
- $$g(x) = \frac{x^3 - 8}{x - 2}$$ $$g(x) = \frac{(x-2)(x^2 + 2x + 4)}{(x-2)}$$ $$g(x) = x^2 + 2x + 4 \quad , x \neq 2$$

(B) Determining Values of Limits

- So then,

$$\begin{aligned} &\lim_{x \to 2} g(x) \\ &= \lim_{x \to 2} \frac{x^3 - 8}{x - 2} \\ &= \lim_{x \to 2} x^2 + 2x + 4 \\ &= (2)^2 + 2(2) + 4 \\ &= 12 \end{aligned}$$

(B) Determining Values of Limits

- We will work with $g(x) = \frac{x^3 - 8}{x - 2}$ and consider the function behaviour at x = 1
- We can express this idea of function behaviour at a point using limit notation as

$$\lim_{x \to 1} \frac{x^3 - 8}{x - 2}$$

(B) Determining Values of Limits

- Determine the following limits.
- Each solution introduces a different “algebra” trick for simplifying the rational expressions
- Verify limit on GDC

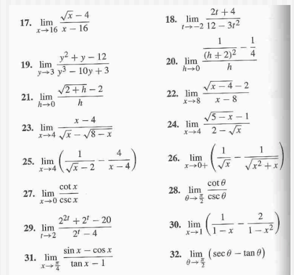
$$\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}$$

$$\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$$

$$\lim_{h\to 0}\frac{(4+h)^3-64}{h}$$

$$\lim_{h\to 0}\frac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h}$$

(B) Determining Values of Limits



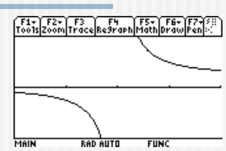
17. $\lim_{x\to 16}\frac{\sqrt{x}-4}{x-16}$
18. $\lim_{t\to -2}\frac{2t+4}{12-3t^2}$
19. $\lim_{y\to 3}\frac{y^2+y-12}{y^3-10y+3}$
20. $\lim_{h\to 0}\frac{\frac{1}{(h+2)^2}-\frac{1}{4}}{h}$
21. $\lim_{h\to 0}\frac{\sqrt{2+h}-2}{h}$
22. $\lim_{x\to 8}\frac{\sqrt{x-4}-2}{x-8}$
23. $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-\sqrt{8-x}}$
24. $\lim_{x\to 4}\frac{\sqrt{5-x}-1}{2-\sqrt{x}}$
25. $\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2}-\frac{4}{x-4}\right)$
26. $\lim_{x\to 0+}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x^2+x}}\right)$
27. $\lim_{x\to 0}\frac{\cot x}{\csc x}$
28. $\lim_{\theta\to \frac{\pi}{2}}\frac{\cot\theta}{\csc\theta}$
29. $\lim_{t\to 2}\frac{2^{2t}+2^t-20}{2^t-4}$
30. $\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right)$
31. $\lim_{x\to \frac{\pi}{4}}\frac{\sin x-\cos x}{\tan x-1}$
32. $\lim_{\theta\to \frac{\pi}{2}}(\sec\theta-\tan\theta)$

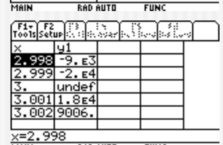
(C) Existence of Limits

- Find $\lim_{x\to 3}\frac{x^2+9}{x-3}$

(C) Existence of Limits

- Find $\lim_{x\to 3}\frac{x^2+9}{x-3}$
- So we try to use some algebra “tricks” as before, but x^2+9 doesn’t factor.
- So we use a ToV, and a graph
- What is the limit in this case?

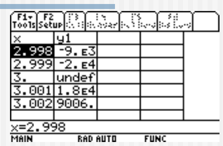


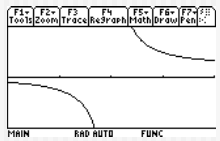


x	y1
2.998	-9.E3
2.999	-2.E4
3.	undef
3.001	1.8E4
3.002	9006.

(C) Existence of Limits

- In this case, both the graph and the table suggest two things:
- (1) as x→3 from the left, g(x) becomes more and more negative
- (2) as x→3 from the right, g(x) becomes more and more positive





(C) Existence of Limits

- So we write these ideas as:
- $\lim_{x\to 3^-}\frac{x^2+9}{x-3}=-\infty$ & $\lim_{x\to 3^+}\frac{x^2+9}{x-3}=\infty$
- Since there is no real number that g(x) approaches, we simply say that this limit does not exist $\lim_{x\to 3}\frac{x^2+9}{x-3}=dne$

(C) Existence of Limits

- Now here is a graph of a function which is defined as

$$f(x)=\begin{cases}2-x & x<2\\(x-3)^2-2 & x\geq 2\end{cases}$$

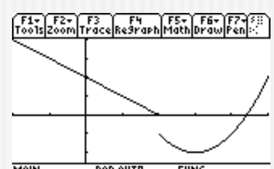
- Find $\lim_{x\to 2}f(x)=$

(C) Existence of Limits

- Now here is a graph of a function which is defined as

$$f(x)=\begin{cases}2-x & x<2\\(x-3)^2-2 & x\geq 2\end{cases}$$

- Find $\lim_{x\to 2}f(x)=$



(C) Existence of Limits

- Now find the limit of this function as *x* approaches 2 where *f*(*x*) is defined as

$$f(x)=\begin{cases} 4-3x & x<2 \\ (2x-3)^2-2 & x\geq 2 \end{cases}$$

- i.e. determine $\lim_{x\to 2} f(x) =$

1/19/17 Mr. Santowski - Calculus & IBHL 25

(C) Existence of Limits

- Now find the limit of this function as *x* approaches 2 where *f*(*x*) is defined as

$$f(x)=\frac{|x-2|}{x-2}$$

- i.e. determine $\lim_{x\to 2} f(x) =$

1/19/17 Mr. Santowski - Calculus & IBHL 26

(C) Existence of Limits

- Now here is an equation of a function which is defined as

$$f(x)=\begin{cases} 3-x & x<2 \\ \frac{2x}{x+2} & x\geq 2 \end{cases}$$

- Find $\lim_{x\to 2} f(x) =$

- Here is an equation of a function which is defined as

$$f(x)=\frac{x^2-x-6}{x+3}$$

- Find $\lim_{x\to -3} f(x) =$

1/19/17 Mr. Santowski - Calculus 27

(C) Existence of Limits

- In considering several of our previous examples, we see the idea of one and two sided limits.
- A **one sided limit** can be a **left handed limit** notated as $\lim_{x\to a^-} f(x)$ which means we approach *x* = *a* from the left (or negative) side
- We also have **right handed limits** which are notated as $\lim_{x\to a^+} f(x)$ which means we approach *x* = *a* from the right (or positive) side

1/19/17 Mr. Santowski - Calculus & IBHL 28

(C) Existence of Limits

- We can make use of the left and right handed limits and now define conditions under which we say a function does not have a limiting y value at a given x value ==> by again considering our various examples above, we can see that some of our functions do not have a limiting y value because as we approach the x value from the right and from the left, we do not reach the same limiting y value.
- Therefore, if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then $\lim_{x \to a} f(x)$ does not exist.

(C) Existence of Limits

2. Sketch each piecewise function below and determine, if it exists, the given limit. If the limit does not exist, provide an explanation.

a) $f(x) = \begin{cases} 2 & , \quad x < 1 \\ 3 & , \quad x = 1 \\ x+1, & \quad x > 1 \end{cases}$ Find $\lim_{x \to 1} f(x)$

b) $f(x) = \begin{cases} 4 - x^2, & -2 < x \leq 2 \\ x - 2 , & x > 2 \end{cases}$ Find $\lim_{x \to 2} f(x)$

c) $f(x) = \begin{cases} |x+2| + 1 , & x < -1 \\ -x + 1 & , -1 \leq x \leq 1 \\ x^2 - 2x + 2 , & x > 1 \end{cases}$ Find $\lim_{x \to 1} f(x)$

(C) Existence of Limits

3. For each function below, determine, if it exists, the given limit. If the limit does not exist, provide an explanation.

a) $f(x) = \begin{Bmatrix} 2x-1, & x \leq -2 \\ -x+2, & x > -2 \end{Bmatrix}$ *Find* $\lim_{x \to -2^+} f(x)$

b) $f(x) = \begin{Bmatrix} -x^2 + 4x - 3, & x < 1 \\ x - 7, & x \geq 1 \end{Bmatrix}$ *Find* $\lim_{x \to 1^-} f(x)$

c) $f(x) = \begin{Bmatrix} x^2 - 2x + 1, & x < -1 \\ -\frac{x}{2} + \frac{7}{2}, & x \geq -1 \end{Bmatrix}$ *Find* $\lim_{x \to -1} f(x)$

d) $f(x) = \begin{Bmatrix} x+3, & x \in (-\infty, 0] \\ -x+2, & x \in (0,2) \\ (x-2)^2, & x \in [2, \infty) \end{Bmatrix}$ *Find* $\lim_{x \to 0} f(x)$ *and* $\lim_{x \to 2} f(x)$

e) $f(x) = \begin{Bmatrix} (x+1)^2 - 1, & -2 \leq x < 0 \\ \frac{5}{4} \sin\left(\frac{\pi x}{2}\right), & 0 \leq x < 2 \\ (x-3)^2 - 1, & 2 \leq x \leq 4 \end{Bmatrix}$ *Find* $\lim_{x \to 2} f(x)$

(D) Limit Laws

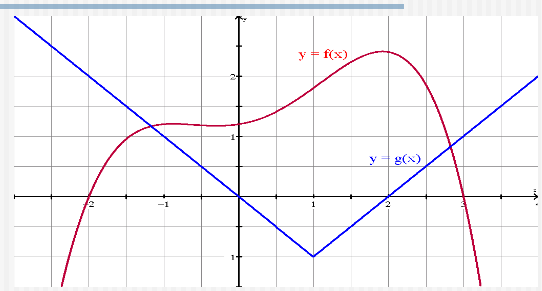
- The limit of a constant function is the constant
- The limit of a sum is the sum of the limits
- The limit of a difference is the difference of the limits
- The limit of a constant times a function is the constant times the limit of the function
- The limit of a product is the product of the limits
- The limit of a quotient is the quotient of the limits (if the limit of the denominator is not 0)
- The limit of a power is the power of the limit
- The limit of a root is the root of the limit

(D) Limit Laws

- Here is a summary of some important limits laws:
- (a) sum/difference rule ➔ lim [f(x) $\pm$ g(x)] = lim f(x) $\pm$ lim g(x)
- (b) product rule ➔ lim [f(x) × g(x)] = lim f(x) × lim g(x)
- (c) quotient rule ➔lim [f(x) ÷ g(x)] = lim f(x) ÷ lim g(x)
- (d) constant multiple rule ➔ lim [kf(x)] = k × lim f(x)
- (e) constant rule ➔ lim (k) = k
- These limits laws are easy to work with, especially when we have rather straight forward polynomial functions

1/19/17 Mr. Santowski - Calculus & IBHL 33

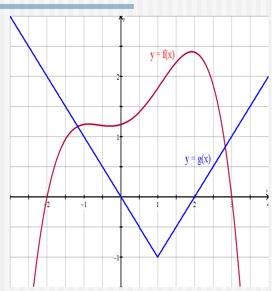
(E) Limit Laws and Graphs



1/19/17 Mr. Santowski - Calculus & IBHL 34

(E) Limit Laws and Graphs

- From the graph on this or the previous page, determine the following limits:
- (1) lim $_{x\to-2}$ [f(x) + g(x)]
- (2) lim $_{x\to-2}$ [(f(x))^2 - g(x)]
- (3) lim $_{x\to-2}$ [f(x) × g(x)]
- (4) lim $_{x\to-2}$ [f(x) ÷ g(x)]
- (5) lim $_{x\to1}$ [f(x) + 5g(x)]
- (6) lim $_{x\to1}$ [½f(x) × (g(x))^3]
- (7) lim $_{x\to2}$ [f(x) ÷ g(x)]
- (8) lim $_{x\to2}$ [g(x) ÷ f(x)]
- (9) lim $_{x\to3}$ [f(x) ÷ g(x)]



1/19/17 Mr. Santowski - Calculus & IBHL 35

(I) "A" Level Investigation

- Research the DELTA-EPSILON definition of a limit
- Tell me what it is and be able to use it
- MAX 2 page hand written report (plus graphs plus algebra) + 2 Q quiz

1/19/17 Mr. Santowski - Calculus & IBHL 36