Lesson 20 - Laws of Logarithms

IB Math HL1 - Santowski

Lesson Objectives

- Understand the rationale behind the "laws of logs"
- Apply the various laws of logarithms in solving equations and simplifying expressions

(A) Properties of Logarithms – Product Law

- Recall the laws for exponents ➔product of powers ➔ $(b^x)(b^y) = b^{(x+y)}$ ➔ so we ADD the exponents when we multiply powers
- For example ➔ $(2^3)(2^5) = 2^{(3+5)}$
- So we have our POWERS ➔ 8 x 32 = 256

(A) Properties of Logarithms – Product Law

- Now, let's consider this from the INVERSE viewpoint
- We have the ADDITION of the exponents
- 3 + 5 = 8
- But recall from our work with logarithms, that the exponents are the OUTPUT of logarithmic functions
- So ➔ 3 + 5 = 8 becomes $\log_2 8 + \log_2 32 = \log_2 256$
- Now, HOW do we get the right side of our equation to equal the left?
- Recall that 8 x 32 = 256
- So $\log_2(8 \times 32) = \log_2 8 + \log_2 32 = \log_2 256$

(A) Properties of Logarithms – Product Law

- So we have our first law ➔ when adding two logarithms, we can simply write this as a single logarithm of the product of the 2 powers

- $\log_a(mn) = \log_a m + \log_a n$
- $\log_a m + \log_a n = \log_a(mn)$

(A) Properties of Logarithms – Formal Proof of Product Law

- Express $\log_a m + \log_a n$ as a single logarithm

- We will let $\log_a m = x$ and $\log_a n = y$
- So $\log_a m + \log_a n$ becomes $x + y$

- But if $\log_a m = x$, then $a^x = m$ and likewise $a^y = n$

- Now take the product $(m)(n) = (a^x)(a^y) = a^{x+y}$
- Rewrite $mn=a^{x+y}$ in log form ➔ $\log_a(mn)=x + y$

- But $x + y = \log_a m + \log_a n$
- So thus $\log_a(mn) = \log_a m + \log_a n$

(B) Properties of Logarithms – Quotient Law

- Recall the laws for exponents ➔Quotient of powers ➔ $(b^x)/(b^y) = b^{(x-y)}$ ➔ so we subtract the exponents when we multiply powers
- For example ➔ $(2^8)/(2^3) = 2^{(8-3)}$
- So we have our POWERS ➔ 256 ÷ 8 = 32

(B) Properties of Logarithms – Quotient Law

- Now, let's consider this from the INVERSE viewpoint
- We have the SUBTRACTION of the exponents
- 8 - 3 = 5
- But recall from our work with logarithms, that the exponents are the OUTPUT of logarithmic functions
- So ➔ 8 - 3 = 5 becomes $\log_2 256 - \log_2 8 = \log_2 32$
- Now, HOW do we get the right side of our equation to equal the left?
- Recall that 256/8 = 32
- So $\log_2(256/8) = \log_2 256 - \log_2 8 = \log_2 32$

(B) Properties of Logarithms – Quotient Law

- So we have our second law ➔ when subtracting two logarithms, we can simply write this as a single logarithm of the quotient of the 2 powers

- $\log_a(m/n) = \log_a m - \log_a n$
- $\log_a m - \log_a n = \log_a(m/n)$

(C) Properties of Logarithms- Logarithms of Powers

- Now work with $\log_3(625) = \log_3(5^4) = x$:

- we can rewrite as $\log_3(5 \times 5 \times 5 \times 5) = x$
- we can rewrite as $\log_3(5) + \log_3(5) + \log_3(5) + \log_3(5) = x$
- We can rewrite as $4\,[\log_3(5)] = 4 \times 1 = 4$

- So we can generalize as $\log_3(5^4) = 4\,[\log_3(5)]$

- So if $\log_3(625) = \log_3(5)^4 = 4 \times \log_3(5)$ ➔ It would suggest a rule of logarithms ➔ $\log_a(b^x) = x \log_a b$

(D) Properties of Logarithms – Logs as Exponents

- Consider the example $3^{\log_3 5} = x$
- Recall that the expression $\log_3(5)$ simply means "the exponent on 3 that gives 5" ➔ let's call that *y*
- So we are then asking you to place that same exponent (the *y*) on the same base of 3
- Therefore taking the exponent that gave us 5 on the base of 3 (*y*) onto a 3 again, must give us the same 5!!!!
- We can demonstrate this algebraically as well

(D) Properties of Logarithms – Logs as Exponents

- Let's take our exponential equation and write it in logarithmic form
- So $3^{\log_3 5} = x$ becomes $\log_3(x) = \log_3(5)$
- Since both sides of our equation have a $\log_3$ then x = 5 as we had tried to reason out in the previous slide
- So we can generalize that $b^{\log_b x} = x$

(E) Summary of Laws

Logs as exponents	$b^{\log_b x} = x$
Product Rule	$\log_a(mn) = \log_a m + \log_a n$
Quotient Rule	$\log_a(m/n) = \log_a(m) – \log_a(n)$
Power Rule	$\text{Log}_a(m^p) = (p) \times (\log_a m)$

(F) Examples

- (i) $\log_3 54 + \log_3(3/2)$
- (ii) $\log_2 144 - \log_2 9$
- (iii) log30 + log(10/3)
- (iv) which has a greater value
  - (a) $\log_3 72 - \log_3 8$ or (b) log500 + log2
- (v) express as a single value
  - (a) $3\log_2 x + 2\log_2 y - 4\log_2 a$
  - (b) $\log_3(x+y) + \log_3(x-y) - (\log_3 x + \log_3 y)$
- (vi) $\log_2(3/4) - \log_2(24)$
- (vii) $(\log_2 5 + \log_2 25.6) - (\log_2 16 + \log_3 9)$

(F) Examples

- Solve for x

$$\log_2 x = 2\log_2 7 + \log_2 3$$
$$\log_2 x = \log_2 11 - \log_2 \sqrt{99}$$
$$\log \sqrt[3]{x} + \log 13 = -\log \frac{1}{91}$$

- Solve for x and verify your solution

$$\log_5 (x+1) + \log_5 3 = 2$$
$$\log_3 (x-2) + \log_3 x = 1$$
$$\log x + \log(x-5) = \log(2x-12)$$

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(F) Examples

- Solve and verify

$$\log_2 x = \frac{1}{3}\log_2 3 + \log_2 \sqrt{3}$$
$$\log_5 (x-1) - \log_5 (x-5) = \log_5 \frac{1}{x+3}$$
$$\log 250 - \log 2 = 3\log \frac{1}{x}$$

- If $a^2 + b^2 = 23ab$, prove that

$$\log\left(\frac{a+b}{5}\right) = \frac{\log a + \log b}{2}$$

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(F) Examples

- Write each single log expression as a sum/difference/product of logs

$$\log \frac{abc^2}{d^3}$$

$$\log \frac{x\sqrt[3]{y}}{z^5}$$

$$\ln \sqrt{\sin x \ln x}$$

(F) Examples

Use the logarithm laws to simplify the following:

(a) $\log_2 xy - \log_2 x^2$

(b) $\log_2 \frac{8x^2}{y} + \log_2 2xy$

(c) $\log_3 9xy^2 - \log_3 27xy$

(d) $\log_4 (xy)^3 - \log_4 xy$

(e) $\log_3 9x^4 - \log_3 (3x)^2$

(F) Examples

Exercise 2. Given $\text{Log}_{10}(5.0) = 0.70$ $\text{Log}_{10}(2.0) = 0.30$ $\text{Log}_{10}(3.0) = 0.48$, without a calculator, determine:

(1) $\text{Log}_{10}(6.0)$
(2) $\text{Log}_{10}(8.0)$
(3) $\text{Log}_{10}(\frac{1}{2})$
(4) $\text{Log}_{10}(15.)$
(5) $\text{Log}_{10}(\frac{2}{3})$
(6) $\text{Log}_{10}(0.40)$
(7) $\text{Log}_{10}(\frac{4}{15})$
(8) $\text{Log}_{10}(\sqrt{5.0})$
(9) $\text{Log}_{10}(\sqrt[4]{3.0})$
(10) $\text{Log}_{10}(0.036)$