Lesson 18 - Solving & Applying Exponential Equations Using Logarithms

IB Math HL1 - Santowski

Fast Five

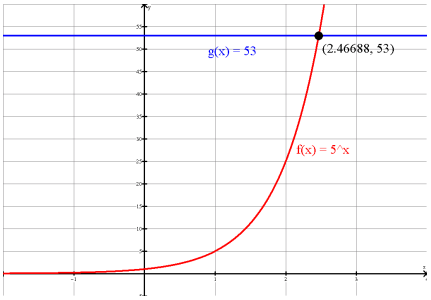
- Solve the following:
- (a) $5^x = 53$
- (b) $\log_3 38 = x$
- (c) Solve $2^x = 7$. HENCE, ALGEBRAICALLY solve $8^{(2x)} = 7^{(1-x)}$

(A) Strategies for Solving Exponential Equations - Guessing

- we have explored a variety of equation solving strategies, namely being able to isolate a variable
- this becomes seemingly impossible for exponential equations like $5^x = 53$
- our earlier strategy was to express both sides of an equation with a common base, (i.e. $2^x = 32$) which we cannot do with the number 53 and the base of 5
- Alternatively, we can simply "guess & check" to find the right exponent on 5 that gives us 53 ➔ we know that $5^2 = 25$ and $5^3 = 125$, so the solution should be somewhere closer to 2 than 3

(B) Strategies for Solving Exponential Equations - Graphing

- Going back the example of $5^x = 53$, we always have the graphing option
- We simply graph $y_1 = 5^x$ and simultaneously graph $y_2 = 53$ and look for an intersection point (2.46688, 53)



(C) Strategies for Solving Exponential Equations - Inverses

- However, one general strategy that we have used previously was to use an "inverse" operation to isolate a variable
- and so now that we know how to "inverse" an exponential expression using logarithms, we will use the same strategy ➔ inverse an exponential using logarithms
- So then if $5^x = 53$, then $\log_5(53) = x$ ➔ but this puts us in the same dilemma as before ➔ we don't know the exponent on 5 that gives 53

(D) Strategies for Solving Exponential Equations - Logarithms

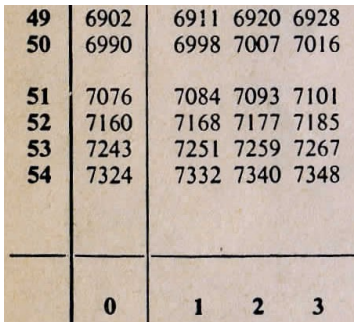
- So we will use the logarithm concept as we apply another logarithm rule ➔ let's simply take a **common logarithm** of each side of the equation ($\log_{10}$) (since our calculators/data tables are programmed to work in base 10)
- Thus, $5^x = 53$ now becomes
- $\log_{10}(5)^x = \log_{10}(53)$
- now what ….. ???

(D) Strategies for Solving Exponential Equations - Logarithms

- So we will use the logarithm concept as we apply another logarithm rule ➔ let's simply take a **common logarithm** of each side of the equation ($\log_{10}$) (since our calculators are programmed to work in base 10)

- Thus, $5^x = 53$ now becomes
- $\log_{10}(5)^x = \log_{10}(53)$
- $(10^{0.6990})^x = 10^{1.7243}$
- So now the bases are the same

- $0.6990x = 1.7243$
- $x = 2.46688$ …..



49	6902	6911	6920	6928
50	6990	6998	7007	7016
51	7076	7084	7093	7101
52	7160	7168	7177	7185
53	7243	7251	7259	7267
54	7324	7332	7340	7348
	0	1	2	3

Properties of Logarithms- Logarithms of Powers

- Use your calculator to evaluate the following and observe a pattern:

- log(2)
- log(4)
- log(8)
- log(16)
- log(32)
- log(64)

Properties of Logarithms- Logarithms of Powers

- Use your calculator to evaluate the following and observe a pattern ➔ but now notice that:

- $\log(2) = \log(2^1)$
- $\log(4) = \log(2^2)$
- $\log(8) = \log(2^3)$
- $\log(16) = \log(2^4)$
- $\log(32) = \log(2^5)$
- $\log(64) = \log(2^6)$

Properties of Logarithms- Logarithms of Powers

- Use your calculator to evaluate the following and observe a pattern ➔ but now notice that ➔ which seems to imply that:

- $\log(2) = \log(2^1) = 1 \times \log(2)$
- $\log(4) = \log(2^2) = 2 \times \log(2)$
- $\log(8) = \log(2^3) = 3 \times \log(2)$
- $\log(16) = \log(2^4) = 4 \times \log(2)$
- $\log(32) = \log(2^5) = 5 \times \log(2)$
- $\log(64) = \log(2^6) = 6 \times \log(2)$

Properties of Logarithms- Logarithms of Powers

- Now now simplify $\log_3(625) = \log_3(5^4) = x$:
- Well, $\log_3(625) = \log_3(5^4)$, so …..
- So we can generalize as $\log_3(5^4) = 4\ [\log_3(5)]$
- So if $\log_3(625) = \log_3(5)^4 = 4 \times \log_3(5)$ ➔ It would suggest a rule of logarithms ➔ $\log_a(b^x) = x \log_a b$

(D) Strategies for Solving Exponential Equations - Logarithms

- So we will use the logarithm concept as we apply another logarithm rule ➔ let's simply take a **common logarithm** of each side of the equation ($\log_{10}$) (since our calculators/data tables are programmed to work in base 10)
- Thus, $5^x = 53$ now becomes
- $\log_{10}(5)^x = \log_{10}(53)$
- now what ….. ???
- $x(\log_{10}(5)) = \log_{10}(53)$
- $x = \log_{10}(53) / \log_{10}(5)$
- $x = 2.4668$ …..

(D) Strategies for Solving Exponential Equations – Natural Logarithms

- In solving $5^x = 53$, we used a common logarithm (log base 10) to solve the equation
- One other common logarithm you will see on your calculator is the natural logarithm (ln which uses a special base of numerical value 2.71828… which is notated by the letter *e* ➔ so loge(x) = ln(x))
- Thus, $\ln 5^x = \ln 53$
- And $x(\ln 5) = \ln 53$
- And $x = \ln 53 \div \ln 5 = 2.46688$ as before

(E) Examples

- Provide algebraic solutions as you solve the following for x. Confirm graphically.
- (c) $3(1.25)^x = 11$
- (d) $4 + 2.5^x = 12$
- (e) $2^{4x+1} = 7^{1-x}$
- (f) $2^{x^2-4} = 9^x$
- (g) $2^{3x+2} = 9$
- (h) $3(2^{2x-1}) = 5^{-x}$
- (i) $2^{4x+1} - 3^x = 0$

53. $2^{3-x} = 565$

54. $8^{-2-x} = 431$

55. $e^{2x} - 4e^x - 5 = 0$

56. $e^{2x} - 5e^x + 6 = 0$

57. $\dfrac{400}{1 + e^{-x}} = 350$

58. $\dfrac{525}{1 + e^{-x}} = 275$

59. $\left(1 + \dfrac{0.10}{12}\right)^{12t} = 2$

60. $\left(16 + \dfrac{0.878}{26}\right)^{3t} = 30$

(F) Applications of Exponential Equations

- EXAMPLE 1: The half-life of radium-226 is 1620 years. After how many years is only 30 mg left if the original sample contained 150 mg?
- Recall the formula for half-life is $N(t) = N_0(2)^{(-t/h)}$ where h refers to the half-life of the substance
- Rework the problem knowing that the formula for ANY exponential application is $y(x) = c(1 + r)^x$.

(F) Applications of Exponential Equations

- The logarithmic function has applications for solving everyday situations:
- Example 2: The value of the Canadian dollar , at a time of inflation, decreases by 10% each year. What is the half-life of the Canadian dollar?

(F) Applications of Exponential Equations

- Ex 3. Find the length of time required for an investment of $1000 to grow to $4,500 at a rate of 9% p.a. compounded quarterly. How does this time change if the investment is compounded continuously?

Example 4

- Mr. S. drinks a cup of coffee at 9:45 am and his coffee contains 150 mg of caffeine. Since the half-life of caffeine for an average adult is 5.5 hours, determine how much caffeine is in Mr. S.'s body at class-time (1:10pm). Then determine how much time passes before I have 30 mg of caffeine in my body.

Example 5

- Dry cleaners use a cleaning fluid that is purified by evaporation and condensation after each cleaning cycle. Every time it is purified, 2% of the fluid is lost
  - An equipment manufacturer claims that after 20 cycles, about two-thirds of the fluid remains. Verify or reject this claim.
  - If the fluid has to be "topped up" when half the original amount remains, after how many cycles should the fluid be topped up?
  - A manufacturer has developed a new process such that two-thirds of the cleaning fluid remains after 40 cycles. What percentage of fluid is lost after each cycle?

Example 6

- You invest $5000 in a stock that grows at a rate of 12% per annum compounded quarterly. The value of the stock is given by the equation $V = 5000(1 + 0.12/4)^{4x}$, or $V = 5000(1.03)^{4x}$ where x is measured in years.
  - Find the value of the stock in 6 years.
  - Find when the stock value is $14,000

Example 7

- The population of a small town was 35,000 in 1980 and in 1990, it was 57,010.

1. Create an algebraic model for the towns population growth.
2. Check your model using the fact that the population was 72800 in 1995.
3. What will the population be in 2010?
4. When will the population be 100,000?

Example 8

- The sales S (in thousands of units) of a new CD burner after it has been on the market for *t* years are given by $S(t) = 100(1 - e^{kt})$. Fifteen thousand units of the new product were sold the first year.
  - Complete the model by solving for *k*.
  - Use your calculator to graph the model.
  - Determine S(3) and interpret.
  - When were 8,000 units sold?

Example 9

- A conservation authority releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a carrying capacity of 1,000 animals and that the growth of the herd will follow the logistic curve , where t is time measured in months.

$$p(t) = \frac{1000}{1 + 9e^{-0.1656t}}$$

  - Graph the function and determine the values of p at which the horizontal asymptotes occur. Interpret the meaning of the asymptotes in the context of the problem.
  - Estimate the population after 5 years.
  - When will the population reach 500?

Example 10

- A coroner is called in to investigate a death. The coroner determines the body's temperature at 9:00 to be 85.7°F and at 11:00 the temperature was 82.8°F. The relationship between time elapsed since death (*t*, in hours) and the body temperature (T in degrees Fahrenheit) is .

$$t(T) = -10 \ln \frac{T - 70}{98.6 - 70}$$

(assume that the body had a normal body temperature of 98.6°F and the room temperature was 70°F). Estimate the time of death of the body.

Example 11

- The model below approximates the length of a home mortgage of $150,000 at 8% in terms of the monthly payment. In the model, *L* is the length of the mortgage in years and *p* is the monthly payments in dollars.

$$L(p) = 12.542 \ln\left(\frac{p}{p-1000}\right), \quad p > 1000$$

  - Use the model to approximate the length of the mortgage when the monthly payments are $1254.68
  - Approximate the total amount paid over the term of the mortgage from (a). What amount of the total is interest costs?
  - If I want to pay off the mortgage in 17 years, how much should my monthly payments be?

Example 12

- In a psychology project on learning theory, a mathematical model for the proportion of correct responses, *P*, after *n* trials was found to be .

$$P(n) = \frac{0.83}{1 + e^{-0.2n}}$$

  - Graph the function
  - Determine the equation(s) of any horizontal asymptotes and interpret in the context of the problem.
  - Determine *P*(0) and *P*(12) and interpret.
  - After how many trials will 60% of the responses be correct?

Example 13

- The demand function for a camera is given by , $p(x) = 500 - 0.5\left(e^{0.004x}\right)$ where *p* is the price of the camera in dollars and *x* is the demand (how many units can be sold at that price).
  - Determine *p*(1600) and interpret in the context of the problem.
  - Determine the demand, x , for a price of (i) $600 and (ii) $400