Lesson 17 - Solving Exponential Equations

IB Math HL - Santowski

Lesson Objectives

- (1) Establish a context for the solutions to exponential equations
- (2) Review & apply strategies for solving exponential eqns:
  - (a) guess and check
  - (b) graphic
  - (c) algebraic
    - (i) rearrange eqn into equivalent bases
    - (ii) isolate parent function and apply inverse

Lesson Objective #1 – Context for Exponential Equations

- (1) Establish a context for the solutions to exponential equations
- (a) population growth
- (b) decay

(A) Context for Equations

- Write and then solve equations that model the following scenarios:
- ex. 1 The model $P(t) = P_o 2^{t/d}$ can be used to model bacterial growth. Given that a bacterial strain doubles every 30 minutes, how much time is required for the bacteria to grow from an initial 100 to 25,600?
- ex 2. The number of bacteria in a culture doubles every 2 hours. The population after 5 hours is 32,000. How many bacteria were there initially?

Lesson Objective #2 – Solving Eqns

- Review & apply strategies for solving exponential eqns:
  - (a) guess and check
  - (b) graphic
  - (c) algebraic
    - (i) rearrange eqn into equivalent bases
    - (ii) isolate parent function and apply inverse

(A) Solving Strategy #1 – Guess and Check

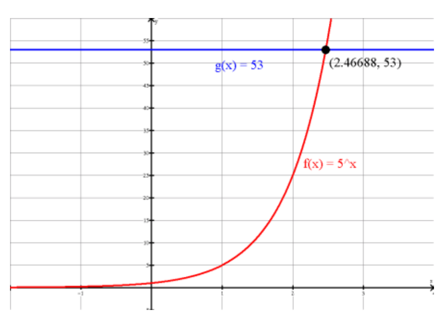
- Solve $5^x = 53$ using a guess and check strategy
- Solve $2^x = 3$ using a guess and check strategy

(A) Solving Strategy #1 – Guess and Check

- Solve $5^x = 53$ using a guess and check strategy
- we can simply "guess & check" to find the right exponent on 5 that gives us 53 ➔ we know that $5^2 = 25$ and $5^3 = 125$, so the solution should be somewhere closer to 2 than 3
- Solve $2^x = 3$ using a guess and check strategy
- we can simply "guess & check" to find the right exponent on 2 that gives us 3 ➔ we know that $2^1 = 2$ and $2^2 = 4$, so the solution should be somewhere between to 1 and 2

(B) Solving Strategy #2 – Graphic Solutions

- Going back the example of $5^x = 53$, we always have the graphing option
- We simply graph $y_1 = 5^x$ and simultaneously graph $y_2 = 53$ and look for an intersection point (2.46688, 53)



(B) Solving Strategy #2 – Graphic Solutions

- Solve the following equations graphically.

- (i) $8^x = 2^{x+1}$
- (ii) $3^x = 53$
- (iii) $2^x = 3$
- (iv) $2^{4x-1} = 3^{1-x}$

(C) Solving Strategy #3 – Algebraic Solutions

- We will work with 2 algebraic strategies for solving exponential equations:
  - (a) Rearrange the equations using various valid algebraic manipulations to either (i) make the bases equivalent or (ii) make the exponents equivalent
  - (b) Isolate the parent exponential function and apply the inverse function to "unexponentiate" the parent function

(D) Algebraic Solving Strategies – Math Principles

- MATH PRINCIPLE ➔ PROPERTIES OF EQUALITIES
- If two powers are equal and they have the same base, then the exponents must be the same
- ex. if $b^x = a^y$ and a = b, then x = y.

- If two powers are equal and they have the same exponents, then the bases must be the same
- ex. if $b^x = a^y$ and x = y, then a = b.

(E) Solving Strategies – Algebraic Solution #1

- This prior observation sets up our general equation solving strategy => get both sides of an equation expressed in the same base

- ex. Solve and verify the following:

- (a) $(\frac{1}{2})^x = 4^{2-x}$ (b) $3^{y+2} = 1/27$
- (c) $(1/16)^{2a-3} = (1/4)^{a+3}$ (d) $3^{2x} = 81$
- (e) $5^{2x-1} = 1/125$ (f) $36^{2x+4} = \sqrt{(1296^x)}$

(E) Solving Strategies – Algebraic Solution #1

- The next couple of examples relate to composed functions ➔quadratic fcns composed with exponential fcns:
- Ex: Let $f(x) = 2^x$ and let $g(x) = x^2 + 2x$, so solve $fog(x) = ½$
- Ex: Let $f(x) = x^2 – x$ and let $g(x) = 2^x$, so solve $fog(x) = 12$

(E) Solving Strategies – Algebraic Solution #1

- The next couple of examples relate to composed functions ➔quadratic fcns composed with exponential fcns:
- Ex: Let $f(x) = 2^x$ and let $g(x) = x^2 + 2x$, so solve $fog(x) = ½$ ➔ i.e. Solve $2^{x^2+2x} = ½$
- Ex: Let $f(x) = x^2 – x$ and let $g(x) = 2^x$, so solve $fog(x) = 12$ ➔ i.e. Solve $2^{2x} - 2^x = 12$

(E) Solving Strategies – Assorted Examples

- Solve the following for x using the most appropriate method:

- (a) $2^x = 8$ (b) $2^x = 1.6$
- (c) $2^x = 11$ (d) $2^x = 32^{2x-2}$
- (e) $2^{4x+1} = 8^{1-x}$ (f) $2^{x^2-4} = 8^x$
- (g) $2^{3x+2} = 9$ (h) $3(2^{2x-1}) = 4^{-x}$
- (i) $2^{4y+1} - 3^y = 0$

Lesson Objective #1 – Context for Exponential Equations

- (1) Establish a context for the solutions to exponential equations

(F) Examples with Applications

- Example 1 ➔ Radioactive materials decay according to the formula $N(t) = N_0(1/2)^{t/h}$ where N_0 is the initial amount, t is the time, and h is the half-life of the chemical, and the (1/2) represents the decay factor. If Radon has a half life of 25 days, how long does it take a 200 mg sample to decay to 12.5 mg?

(F) Examples with Applications

- Example 2 ➔ A financial investment grows at a rate of 6%/a. How much time is required for the investment to double in value?

- Example 3 ➔ A financial investment grows at a rate of 6%/a but is compounded monthly. How much time is required for the investment to double in value?

(F) Examples with Applications

- ex 1. Mr. S. drinks a cup of coffee at 9:45 am and his coffee contains 150 mg of caffeine. Since the half-life of caffeine for an average adult is 5.5 hours, determine how much caffeine is in Mr. S.'s body at class-time (1:10pm). Then determine how much time passes before I have 30 mg of caffeine in my body.
- ex 2. The value of the Canadian dollar , at a time of inflation, decreases by 10% each year. What is the half-life of the Canadian dollar?

(F) Examples with Applications

- ex 3. The half-life of radium-226 is 1620 a. Starting with a sample of 120 mg, after how many years is only 40 mg left?
- ex 4. Find the length of time required for an investment of $1000 to grow to $4,500 at a rate of 9% p.a. compounded quarterly.

(F) Examples with Applications

- ex 5. Dry cleaners use a cleaning fluid that is purified by evaporation and condensation after each cleaning cycle. Every time it is purified, 2% of the fluid is lost
- (a) An equipment manufacturer claims that after 20 cycles, about two-thirds of the fluid remains. Verify or reject this claim.
- (b) If the fluid has to be "topped up" when half the original amount remains, after how many cycles should the fluid be topped up?
- (c) A manufacturer has developed a new process such that two-thirds of the cleaning fluid remains after 40 cycles. What percentage of fluid is lost after each cycle?

(F) Examples with Applications

- Ex 1. An investment of $1,000 grows at a rate of 5% p.a. compounded annually. Determine the first 5 terms of a geometric sequence that represents the value of the investment at the end of each compounding period.
- Ex 2. Find the length of time required for an investment of $1000 to grow to $4,500 at a rate of 9% p.a. compounded quarterly.

(F) Examples with Applications

- Write and then solve equations that model the following scenarios:
- Ex 1. 320 mg of iodine-131 is stored in a lab for 40d. At the end of this period, only 10 mg remains.
  - (a) What is the half-life of I-131?
  - (b) How much I-131 remains after 145 d?
  - (c) When will the I-131 remaining be 0.125 mg?
- Ex 2. Health officials found traces of Radium F beneath P044. After 69 d, they noticed that a certain amount of the substance had decayed to $1/\sqrt{2}$ of its original mass. Determine the half-life of Radium F

Lesson 17 – Introducing and Applying Base *e*.

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FAST FIVE

- Go to the following DESMOS interactive graph from the lesson notes page and play with it to the end that you can explain what is going on.
- CONTEXT ➔ $1000 invested at 6% p.a compounded *n* times per year.

(A) FAST FIVE

- 1. Given the geometric series defined as $S_n = \sum_{k=0}^{n} \left(\frac{1}{k!} \right)$
- (a) List the first 5 terms of the series
- (b) Find the first five partial sums
- (c) evaluate the series if n = 20
- (d) evaluate the series as n ➔ ∞
- 2. Evaluate $f(x) = \left(1 + \frac{1}{x}\right)^x$ for $x = \{1,2,5,15,50\}$

Lesson Objectives

- (1) Investigate a new base to use in exponential applications
- (2) Understand WHEN the use of base e is appropriate
- (3) Apply base e in word problems and equations

Lesson Objective #1

- (1) Investigate a new base to use in exponential applications
- GIVEN: the formula for working with compound interest ➔

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

- Determine the value after 2 years of a $1000 investment under the following compounding conditions:

(A) Working with Compounding Interest

- Determine the value after 2 years of a $1000 investment under the following investing conditions:

- (a) Simple interest of 10% p.a
- (b) Compound interest of 10% pa compounded annually
- (c) Compound interest of 10% pa compounded semi-annually
- (d) Compound interest of 10% pa compounded quarterly
- (e) Compound interest of 10% pa compounded daily
- (f) Compound interest of 10% pa compounded n times per year

(B) Introducing Base *e*

- Take $1000 and let it grow at a rate of 10% p.a. Then determine value of the $1000 after 2 years under the following compounding conditions:

- (i) compounded annually $=1000(1 + .1/1)^{(2x1)} = 1210$

- (ii) compounded quarterly $=1000(1 + 0.1/4)^{(2x4)} = 1218.40$

- (iii) compounded daily $=1000(1 + 0.1/365)^{(2x365)} = 1221.37$

- (iv) compounded *n* times per year $=1000(1+0.1/n)^{(2xn)} = ????$

(B) Introducing Base *e*

- So we have the expression 1000(1 + 0.1/n)$^{(2n)}$
- Now what happens as we increase the number of times we compound per annum ⇒ i.e. n →∞ ?? (that is … come to the point of **compounding continuously**)
- So we get the idea of an "end behaviour again:

$$1000\times\left(1+\frac{0.1}{n}\right)^{(2\times n)} \quad \text{as } n\to\infty$$

(B) Introducing Base *e*

- Now let's rearrange our function
- use a simple substitution ➔ let 0.1/n = 1/x
- Therefore, 0.1x = n ➔ so then $1000\times\left(1+\frac{0.1}{n}\right)^{(2\times n)}$ as $n\to\infty$

becomes $1000\times\left(1+\frac{1}{x}\right)^{(x\times 0.1\times 2)}$ as $x\to\infty$

- Which simplifies to $1000\times\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2}$

(B) Introducing Base *e*

- So we see a special end behaviour occurring:

$$1000 \times \left(\left(1 + \frac{1}{x} \right)^{x} \right)^{0.1 \times 2} \quad \text{as } x \to \infty$$

- Now we will isolate the "base" of $\left(1 + \frac{1}{x} \right)^{x}$
- We can evaluate the limit a number of ways ➔ graphing or a table of values.

(B) Introducing Base *e*

- So we see a special "end behaviour" occurring:

$$1000 \times \left(\left(1 + \frac{1}{x} \right)^{x} \right)^{0.1 \times 2} \quad \text{as } x \to \infty$$

- We can evaluate the "base" $\left(1 + \frac{1}{x} \right)^{x}$ by graphing or a table of values.
- In either case, $e = \left(1 + \frac{1}{x} \right)^{x} \quad \text{as } x \to \infty$
- where *e* is the natural base of the exponential function

(B) Introducing Base *e*

- So our original formula $1000\times\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2}$ as $x \to \infty$ now becomes A = 1000$e^{0.1x2}$ where the 0.1 was the interest rate, 2 was the length of the investment (2 years) and $1000 was the original investment (so A = Pe^{rt}) ➔ so our value becomes $1221.40
- And our general equation can be written as A = Pe^{rt} where P is the original amount, r is the annual growth rate and t is the length of time in years

Lesson Objective #2

- (2) Understand WHEN the use of base e is appropriate
- Recall our original question ➔ Determine the value after 2 years of a $1000 investment under the following investing conditions:
- (a) Compound interest of 10% pa compounded annually
- (b) Compound interest of 10% pa compounded semi-annually
- (c) Compound interest of 10% pa compounded quarterly
- (d) Compound interest of 10% pa compounded daily
- All these examples illustrate **DISCRETE** changes rather than **CONTINUOUS** changes.

Lesson Objective #2

- (2) Understand WHEN the use of base e is appropriate

- So our original formula $1000\times\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2}$ as $x\to\infty$ now becomes A = 1000e$^{0.1x2}$ where the 0.1 was the annual interest rate, 2 was the length of the investment (2 years) and $1000 was the original investment ➔ BUT RECALL WHY we use "n" and what it represents …….

- Note that in this example, the growth happens ***continuously*** (i.e the idea that n→∞)

Lesson Objective #3

- (3) Apply base ***e*** in word problems and equations

(C) Working With Exponential Equations in Base **e**

- (i) Solve the following equations:

$(i)\ e^{x^2-x}=e^2$ $\quad (ii)\ \left(e^x\right)^2=\sqrt{e^{x+2}}$

$(ii)\ e^{-x^2}=\left(\frac{1}{e}\right)^x$ $\quad (iv)\ e^{2x-1}=\frac{1}{e^{3x+1}}$

$(v)\ e^x=4$ $\quad (vi)\ e^x=-5$

$(vii)\ e^x=1-x$ $\quad (viii)\ e^{2x}+6=5e^x$

(C) Working with A = Pe^rt

- So our formula for situations featuring continuous change becomes A = Pe^{rt} ➔ P represents an initial amount, r the annual growth/decay rate and t the number of years
- In the formula, if r > 0, we have exponential growth and if r < 0, we have exponential decay

(C) Examples

- (i) I invest \$10,000 in a funding yielding 12% p.a. compounded continuously.
  - (a) Find the value of the investment after 5 years.
  - (b) How long does it take for the investment to triple in value?
- (ii) The population of the USA can be modeled by the eqn $P(t) = 227e^{0.0093t}$, where P is population in millions and t is time in years since 1980
  - (a) What is the annual growth rate?
  - (b) What is the predicted population in 2015?
  - (c) What assumptions are being made in question (b)?
  - (d) When will the population reach 500 million?

(C) Examples

- A population starts with 500 viruses that grows to a population of 600 viruses in 2 days.
- (a) Assuming LINEAR GROWTH, write a linear model
  ($u_n = u_1 + (n-1)d$) for population growth
- (b) Assuming DISCRETE EXPONENTIAL GROWTH, write an exponential model ($u_n = u_1 r^{n-1}$) for population growth
- (c) Assuming CONTINUOUS EXPONENTIAL GROWTH, write an exponential model ($A = A_0e^{rt}$) for population growth.
- Use your models to predict the number of viruses in one month.
- Explain WHY it is important to work with APPROPRIATE and ACCURATE models, given the recent Ebola outbreak in West Africa

(C) Examples

- (iii) A certain bacteria grows according to the formula $A(t) = 5000e^{0.4055t}$, where t is time in hours.
  - (a) What will the population be in 8 hours
  - (b) When will the population reach 1,000,000

- (iv) The function $P(t) = 1 - e^{-0.0479t}$ gives the percentage of the population that has seen a new TV show t weeks after it goes on the air.
  - (a) What percentage of people have seen the show after 24 weeks?
  - (b) Approximately, when will 90% of the people have seen the show?
  - (c) What happens to P(t) as t gets infinitely large? Why? Is this reasonable?

(F) Examples with Applications

- Two populations of bacteria are growing at different rates. Their populations at time *t* are given by $P_1(t) = 5^{t+2}$ and $P_2(t) = e^{2t}$ respectively. At what time are the populations the same?